NONLINEAR OSCILLATIONS OF MODERATELY THICK STRUCTURAL ELEMENTS

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DOCTOR OF PHILOSOPHY

BY
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to the

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INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JULY 1970

TO MY

PARENTS, WIFE, AND CHILDREN

ME - 1970 - D - SIN - NON

CERTIFICATE

This is to certify that the thesis entitled "Nonlinear Oscillations of Moderately Thick Structural Elements" by Pashupati Nath Singh, for the award of the Degree of Doctor of Philosophy, of the Indian Institute of Technology, Kanpur is record of bonafide research work carried out by him under our supervision and guidance. The thesis work, in our opinion reached the standard fulfilling the requirements for the Doctor of Philosophy Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

of the

Ph.D. Dissertation

on

NONLINEAR OSCILLATIONS OF MODERATELY THICK STRUCTURAL ELEMENTS

by

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Large amplitude oscillations of straight beam and shallow arch, rectangular and circular plates, a doubly curved shallow shell, and a spherical cap have been studied in this work. Thin and moderately thick structural elements have been analysed. Material of these elements have been assumed to be isotropic, homogeneous and linearly elastic. The elongations and shears are assumed to be small but rotations are moderately large. Nonlinear strain-displacement relations based on Kármán's assumptions have been used. The effects of transverse shear and rotatory inertia have been taken into account by the so-called method of internal constraints. Either the equations of motion have been derived by the use of Hamilton's principle in the way indicated by Reissner or

the equations of motion of the elasticity theory have been employed. Fixed inertial reference frame has been taken and Lagrangian description of motion has been employed.

By assuming space modes and applying Galerkin technique the equations of motion have been reduced to ordinary differential equations in which time is the independent variable. The resulting equations have been combined into one higher order equation. Fourth and higher order derivatives of time have been neglected to get one second order equation in time. These have negligible effect on the fundamental period of vibration, which has been obtained in the present work."

The resulting equation has been integrated numerically to obtain the period of flexural vibration. Relationship between amplitude and period has been studied. Results obtained have been compared, for limiting cases, with available results and in general the agreement is very satisfactory. Effects like dynamic buckling, transition from a plate to a shallow shell etc. have been studied.

While the method is general, the study has been limited to structures with hinged ends and only the fundamental period of vibration has been obtained.

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NOMENCLATURE

A_R	-	amplitude ratio (amplitude of vibration/h)
a	-	length of beam, length of the plan of shallow arch, one side of a rectangular plate, one side of the plan of a Doubly Curved shallow shell, radius of circular disk, or the radius of the base of spherical cap
B	-	b/a
b	-	one side of rectangular plate or that of the plan of Doubly Curved shallow shell ($b \leq a$)
c	-	initial maximum rise of middle surface of Doubly Curved shell or arch
D	-	$\frac{Eh^3}{12(1-\nu^2)}$
E	-	Young's modulus of elasticity
G	-	shear modulus; $G = \frac{E}{2(1+\nu)}$
H	-	$\frac{h}{a}$
H^*	-	initial maximum rise of spherical cap
h	-	thickness of the structural elements
I_0, I_1	-	Modified Bessel function of the first kind of order zero and one
J_0, J_1	-	Bessel function of the first kind of order zero and one
k	-	c/h

M_x, M_y - bending moments per unit length

M_{xy}, M_{yx} - twisting moments per unit length

N_x, N_y - in-plane normal forces per unit length

N_{xy}, N_{yx} - shearing forces per unit length

R - radius of curvature of shallow spherical cap

R^* - $\frac{a}{R}$

S - surface

T - non-dimensional time $(= t.\sqrt{\frac{E}{\rho a^2}})$

T_C - linear time period by classical theory (non-dimensional)

T_L - linear time period with effects of transverse shear and rotatory inertia (non-dimensional)

T_N - nonlinear time period with effects of transverse shear and rotatory inertia (non-dimensional)

t - time

u, v, w - components of displacement along rectangular cartesian coordinates ; x, y, z

u_r - radial displacement on the middle surface in case of shallow spherical cap or circular disk

u^z - radial displacement at a distance z from middle surface in case of shallow spherical cap or circular disk

V - volume

$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz},$

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ - normal and shear strains

ϵ_{rr} - strain in radial direction

$\epsilon_{\theta\theta}$	-	strain in transverse direction
ϵ^{o}_{rr}	-	ϵ_{rr} on the middle surface
$\epsilon^{o}_{\theta\theta}$	-	$\epsilon_{\theta\theta}$ on the middle surface
ν	-	Poisson's radio
ρ	-	mass per unit volume
σ_{xx}, σ_{yy}, σ_{zz}, τ_{xy}, τ_{xz}, τ_{yz}	-	normal and shear stresses
$\doteq$	-	denotes "approximately equal to"
$(\dot{\ })$	-	denotes partial or total differentiation with respect to t
$(\ )'$	-	denotes partial or total differentiation with respect to T
$(\ )_{,x}$	-	denotes partial or total differentiation with respect to x (similarly for other independent variables)

CHAPTER I

INTRODUCTION

1.1 GENERAL

Beams, plates, and shells are common structural elements and the importance of accurate analysis of their dynamic behaviour need not be emphasised. It may very well be said that the most important structural elements of the modern airplane, missile, booster, and re-entry vehicle comprise of shells, plates, and beams. When these elements are subjected to severe operational conditions, the amplitudes need not be small. Since the linear theories are valid only for small deflections, it is necessary to employ nonlinear theories which allow for finite deflections.

In elementary analysis the effects of transverse shear and rotatory inertia are neglected. These have appreciable effects when the members are not very thin and also when there are large deformations. Also, the effects become more pronounced in higher modes of vibration.

The expressions "transverse shear" and "rotatory inertia", which occur frequently in the present work are more or less self-explanatory. The normal to the middle surface of the structural element before deformation may not remain normal to the middle surface of the element

after deformation. This is ignored in elementary analysis. Apart from this, the elements of a structural member perform not only a translatory but also a rotatory motion during flexural vibrations. The latter is ignored in the classical theories. S. Timoshenko is credited with having developed the governing equations which include the effects of transverse shear and rotatory inertia in the case of a beam, although both the effects were discussed as early as 1859 by M. Bresse*. There are many ways to include these effects in the formulation and one such method, the so-called method of "internal constraints" will be employed here.

Approximate solutions of nonlinear equations become necessary since most of them cannot be integrated explicitly in terms of known functions. To be meaningful these solutions must be accurate within certain acceptable limit and must accommodate the characteristic features of the physical problem.

In the problems that have been considered, elongations and shears are assumed to be small but rotations are moderately large. This leads to nonlinear strain-displacement relations. On the other hand the

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* Cours de Mécanique Appliqués, Mallet-Bachelier, Paris, 1859, p. 126.

strains are assumed to maintain linear relationships with stresses. Thus the problem becomes geometrically nonlinear but remains physically (or materially) linear. The study of the effect of material nonlinearity is a separate class of problems in itself and has not been considered here. The structural material has also been assumed to be isotropic and homogeneous. It may be pointed out that in the present age, there is no dearth of materials with the above properties. All the structural elements considered have been assumed to be of uniform thickness.

Free flexural vibrations of thin and moderately thick structural elements have been studied. Fundamental period (corresponding to the fundamental frequency), which is of primary concern to structural engineers, has been obtained. Effects of damping and of coupling of modes have been neglected.* The structural elements that have been analysed are : straight beam and shallow arch, rectangular and circular plates, a doubly-curved shallow shell on rectangular plan, and a spherical cap. These are the commonly used elements in engineering design that are subjected to bending. Results obtained have been compared, in limiting cases, with known solutions and in

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* J.D. Ray and C.W. Bert (Ref. 1) have shown by experimental investigation on a beam that these effects are negligible.

general the agreement is very satisfactory. By computing the values of fundamental period in a few cases it has been amply demonstrated that by an incorrect but frequently employed application of Galerkin technique, the values computed come out to be erroneous. This fact for the case of a sandwich plate has been pointed out by Yi-Yuan Yu and Jai-Lue Lai*.

1.2 LITERATURE SURVEY

A survey of literature on free vibration problems in beams, plates, and shells reveals that either large deflection problems neglecting the effects of transverse shear and rotatory inertia or small deflection problems taking these effects into account have been studied.

The linear analysis of free vibrations of beams where transverse shear and rotatory inertia effects are considered are dealt with extensively in literature (Refs. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 etc.). Large amplitude vibrations of beams have been discussed by a number of authors (Refs.12, 13, 14, 15, 16, 17 etc.).

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* "Application of Galerkin's Method to the Dynamic Analysis of Structures", AIAA Journal, April 1967, pp. 792-795.

Eringen[12*] derived the governing equations for flexural vibrations of thin elastic beams when the displacements are finite but the strains are small. He assumed the motion to be extensional which implied the presence of a resultant normal force acting on the cross-section of the beam and obtained very complicated governing equations. Wagner[13] formulated a system of governing equations for a similar problem using energy techniques and disregarding the resultant normal force in the cross-sections of the beam. Libai[14] derived a system of governing equations of motion for the above problem using a convected coordinate system. He made an additional assumption that the component of the instantaneous velocity of a point on the middle surface of the beam could, under certain circumstances, be neglected as compared to the component normal to the surface. By using a potential function and assuming the extensional strains to be very small compared to rotations, he obtained much simpler equations than those given by Eringen. Woodall[15] also solved the problem of finite amplitude, free, planar oscillations of a thin elastic beam assuming the motion to be inextensional and allowing

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* The raised numbers indicate references listed at the end of the thesis.

the existence of a resultant normal force acting on each cross-section of the beam. For a simply supported beam a finite difference, Galerkin, and (regular) perturbation solutions have been compared. Ray and Bert[1] have investigated both theoretically and experimentally the large amplitude vibrations of a flexible beam with pinned ends. They have discussed the extent of validity of various assumptions that are usually made while analysing beam vibration problems. Chu and Herrmann[17] and Wu and Vinson[16] have investigated the beam problem as a special case of the plate problem.

Large amplitude vibration of rectangular plates have been studied by a number of authors (Refs. 17, 16, 18 etc.). Chu and Herrmann [17] made Karman's type of assumptions for strains and derived the governing equations in terms of components of displacement for thin plates. They used perturbation technique to solve the equations. Wu and Vinson[16] extended a part of the work by Nash and Modeer[18], who had investigated thin plate vibrations in the so-called Berger's field (they neglected the second invariant of strain). Using Hamilton's principle in the way indicated by Reissner, Wu and Vinson obtained the equations of motion for large amplitude oscillations of transversely isotropic

plates. In the solution they have included the effect of transverse shear but neglected the effect of rotatory inertia. Chauhan and Ashwell[19] studied the vibration of a thin plate as a special case of a shallow shell. They investigated small and large amplitude oscillations of a doubly curved shell on square plan. They used Rayleigh-Ritz method for the linear problem and attempted an approximate analysis for the nonlinear problem by making energy consideration and finally solving the nonlinear equation by numerical technique. They have neglected the effects of transverse shear and rotatory inertia.

Vibration of shallow arches have been studied extensively since they are the simplest structural elements about which detailed studies may be feasible and which have many common features with shells.

Snap-through buckling of curved elements such as arches and shallow shells has received considerable interest in elastic stability. If the applied loading causes the curvature to decrease, the element may snap-through to a new equilibrium configuration. The new equilibrium of buckled configuration is characterised by having a curvature opposite to the undeformed shape. The dynamic analog of this problem has been of

considerable interest. Studies in this field include loading which is suddenly applied and maintained (Refs. 20, 21⁺), periodic vibrations superimposed on a static loading (Ref. 23), and impulsive loading (Refs. 24, 25, 26, 27). In these analyses, the spatial form of the displacement function is assumed. Approximate solutions are then obtained by using variational principles. The essential difference in these studies is in the choice of the spatial form.

Static stability of shallow arches has been studied in a very complete manner by Fung and Kaplan[28]. The corresponding dynamic problem was studied by Hoff and Bruce[20] who have analysed various stability problems by using, in most cases, a two mode representation for the response. Hsu[29] considered the dynamic stability of shallow arches against snap-through when subjected to impulsive loads. He took all modes in the response into account and investigated the effects of various parameters. Later Hsu[30] and Hsu et. al.[31] investigated the stability of shallow arches against snap-through under timewise step loads. Lock[21] and Humphreys[26] used a digital and an analog computer respectively to calculate the responses of a shallow arch in order to establish stability criteria. Lock used two modes while

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⁺ Earlier this work had appeared as a report (Ref.22).

Humphreys used six modes in his analysis.

Vibrations of plates and cylindrical shells have been investigated quite extensively whereas results on vibrations of spherical shells are less comprehensive. Federhofer[32] and Reissner[33] have obtained approximate values of fundamental frequencies for the linear axisymmetric vibrations of a clamped spherical cap by the use of Ritz method. Later Reissner[34], in dealing with fundamental frequency of transverse vibrations of shallow spherical shells, justified the omission of longitudinal inertia terms from the governing equations of such vibrations. His results for the fundamental frequency are in good agreement with those obtained more recently by Kalnins[35] from the exact solution of the shallow shell equations. The effect of thickness-shear deformation in the linear problem was taken into account by Kalnins[36] for a shallow spherical shell and by Deresiewicz and Mindlin[37] and Deresiewicz[38] for circular discs. Also, Reissner[39] solved the problem of axisymmetric vibrations of circular plates of uniform thickness and included the effects of transverse shear and rotatory inertia. However, no numerical results were given. Koplik and Yu[40*] investigated vibrations of homogeneous and sandwich spherical caps with clamped edges and

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* The method is based on a generalised Hamilton's principle and in particular, the associated variational equation of motion for nonlinear case that were used for plates and shells by Yu (Refs. 41, 42).

included the effects of transverse shear and rotatory inertia. They have emphasised the transition from a circular disk to a shallow shell - the two problems that had been treated so far in a more or less isolated manner. Humphreys and Bodner[24], Budiansky and Roth[27], Archer and Lange[43], Simitses[44], Lock et. al.[45], and Huang[46] also investigated dynamic buckling of shallow spherical shells under certain kinds of loading.

More recently Grossman, Koplik, and Yu[47] solved the geometrically nonlinear vibration problem of a shallow spherical cap for various boundary conditions. The formulation is general - the effects of transverse shear and rotatory inertia are included. Thereafter these effects have been neglected and the equations turn out to be the same as those obtained by Connor[48]. These simplified equations have been solved and the effects of curvature, amplitude, and edge supports on the vibration have been studied. Here also the emphasis is on transition from a slightly curved plate to a shallow shell. This phenomenon is investigated by obtaining the ratio of initial rise to thickness for which the type of nonlinearity changes from hardening to softening. It may be mentioned that nonlinearity in the case of flexural vibration of a circular ring

(as in the case of a shallow shell) is of softening type. This was observed, both theoretically and experimentally for a circular ring by Evensen[49]. Subsequently he confirmed experimentally (Ref.50) that nonlinearity in the case of thin cylindrical shell is also of softening instead of hardening type. Earlier Yu[51] had reduced the problem of nonlinear vibration of cylindrical shells to a nonlinear equation in time, which contained both quadratic and cubic nonlinear terms (the former being explicitly associated with curvature). However, by improper choice of displacement, the effect of quadratic nonlinearity was missed and erroneously the nonlinearity was found to be of hardening type. A better form of displacement has been used in Ref.47 to retain the effect of quadratic nonlinearity also. However, no difference between positive (outwards) and negative(inwards) amplitudes of vibration of the spherical cap is made in the above work. Consequently, the phenomenon of dynamic buckling is not brought out.

1.3 OBJECT AND SCOPE

In the present work, a method has been suggested to obtain the natural frequencies for large amplitude oscillations of some moderately thick common structural elements like beams, plates and shells. The effects of transverse shear and rotatory inertia have been included. The material is assumed to be linearly elastic but the strain-displacement relations are taken to be nonlinear. A fixed inertial reference frame and a Lagrangian description of motion have been employed. The problems of a rectangular plate, a doubly curved shallow shell on rectangular plan, and a spherical cap have been analysed. The results are specialised for straight beam, a shallow arch, and a circular disk. Thickness of the elements has been taken to be uniform. In all cases except that of a spherical cap, the equations of motion have been derived using Hamilton's principle in the manner indicated by Reissner. For a spherical cap, the equations of motion of the elasticity theory are used. However, the final results obtained for this case are compared with the results obtained from equations derived by a variational approach. The agreement for the limiting case is good.

In Chapter II, the problems of the above structural elements have been formulated. The equations of motion have been obtained in terms of components of displacement and those of rotation. Assuming approximate mode shapes that satisfy the boundary conditions and using well-known Galerkin technique, ordinary nonlinear differential equations with time as independent variable have been derived. These equations have been combined into one higher order equation. Neglecting fourth and higher order derivatives, which have negligible effect on the fundamental period, one second order nonlinear equation in time has been obtained. This has been integrated numerically. Although the method is general, the computations are restricted to obtaining fundamental period of elements with hinged ends.

Chapter III contains the results computed for various thicknesses of elements. The results are discussed in detail and are compared in limiting cases with available results. In general the agreement is very satisfactory.

Chapter IV lists the conclusions derived from the results computed in this work.

Recommendations for further work are given in Chapter V.

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CHAPTER II

FORMULATION

2.1 THE CASE OF A RECTANGULAR PLATE

In this section, the governing equations are derived for a rectangular plate (see Fig.1). These are then specialised for the case of a straight beam. Reissner's variational theorem (Ref.52) extended for the dynamic case may be written as :

$$\delta \int_{t_1}^{t_2} \left[\iiint_V F.dV - \iint_{S_1} (\bar{p}_x.u + \bar{p}_y.v + \bar{p}_z.w)dS - \iiint_V \rho/2(\dot{u}^2 + \dot{v}^2 + \dot{w}^2)dV \right] dt = 0 \qquad (2.1)$$

where t_1 to t_2 is an arbitrary interval of time

$$F = \sigma_{xx}.\epsilon_{xx} + \sigma_{yy}.\epsilon_{yy} + \sigma_{zz}.\epsilon_{zz} + \tau_{xy}.\gamma_{xy} + \tau_{xz}.\gamma_{xz} + \tau_{yz}.\gamma_{yz} - \bar{W}$$

$\epsilon_{xx} = \bar{W}$ (σ_{xx}, -----)$_{,\sigma xx}$ etc.

S_1 = that part of the surface where stress boundary conditions are prescribed

$\bar{p}_x, \bar{p}_y, \bar{p}_z$ = Components of surface tractions prescribed over S_1

Equation (2.1) may be reduced to the form (see, for example Ref.53) :

$$\int_{t_1}^{t_2} \left[\delta \left\{ \iiint_V F.dV - \iint_{S_1} (\bar{p}_x u + \bar{p}_y v + \bar{p}_z w) dS \right\} \right.$$

$$\left. + \rho \iiint_V (\ddot{u}.\delta u + \ddot{v}\,\delta v + \ddot{w}.\delta w) dV \right] dt = 0 \qquad (2.2)$$

Since the interval t_1 to t_2 is arbitrary :

$$\delta \left\{ \iiint_V F.dV - \iint_{S_1} (\bar{p}_x u + \bar{p}_y v + \bar{p}_z w) dS \right\}$$

$$+ \rho \iiint_V (\ddot{u}.\delta u + \ddot{v}.\delta v + \ddot{w}.\delta w) dV = 0 \qquad (2.3)$$

In the course of this work finite deformations have been assumed. Elongations and shears are small but rotations are moderately large. Karman's type assumptions (see, for example Ref. 54) may therefore be made :

$$\epsilon_{xx} = u_{,x} + 1/2.(w_{,x})^2$$

$$\epsilon_{yy} = v_{,y} + 1/2.(w_{,y})^2$$

$$\epsilon_{zz} = w_{,z} + 1/2.(w_{,z})^2 \qquad (2.4)$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x}.w_{,y}$$

$$\gamma_{xz} = u_{,z} + w_{,x} + w_{,x}.w_{,z}$$

$$\gamma_{yz} = v_{,z} + w_{,y} + w_{,y}.w_{,z}$$

To include the effects of transverse shear and rotatory inertia in the governing equations, the so-called method of "internal constraints" will be used here; that is, the components of elastic displacement must comply with special equations of constraint (Ref. 55). The components of displacement are then taken in the form :

$$
\begin{aligned}
u(x,y,z,t) &= u_o(x,y,t) + z.\alpha(x,y,t) \\
v(x,y,z,t) &= v_o(x,y,t) + z.\beta(x,y,t) \\
w(x,y,z,t) &= w(x,y,t)
\end{aligned}
\tag{2.5}
$$

Therefore,

$$
\begin{aligned}
\epsilon_{xx} &= u_{o,x} + z.\alpha_{,x} + 1/2.(w_{,x})^2 \\
\epsilon_{yy} &= v_{o,y} + z.\beta_{,y} + 1/2.(w_{,y})^2 \\
\epsilon_{zz} &= 0 \\
\gamma_{xy} &= u_{o,y} + v_{o,x} + z\,(\alpha_{,y} + \beta_{,x}) + w_{,x}.w_{,y} \\
\gamma_{xz} &= \alpha + w_{,x} \\
\gamma_{yz} &= \beta + w_{,y}
\end{aligned}
\tag{2.6}
$$

The stresses are taken in the form (see Fig.2) :

$$\sigma_{xx} = \frac{N_x}{h} + \frac{12.z.M_x}{h^3} \tag{2.7.1}$$

$$\sigma_{yy} = \frac{N_y}{h} + \frac{12.z.M_y}{h^3} \tag{2.7.2}$$

$$\sigma^{*}_{zz} \doteq 0 \tag{2.7.3}$$

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* σ_{zz} has been taken to be zero for the case of free vibration.

$$\tau_{xy} = \frac{N_{xy}}{h} + \frac{12.z.M_{xy}}{h^3} \qquad (2.7.4)$$

$$\tau_{xz} = \frac{3Q_x}{2h}\left\{1 - \left(\frac{2z}{h}\right)^2\right\} \qquad (2.7.5)$$

$$\tau_{yz} = \frac{3Q_y}{2h}\left\{1 - \left(\frac{2z}{h}\right)^2\right\} \qquad (2.7.6)$$

The analysis is restricted to the case when S_1=0. Substituting eqs.(2.6) and (2.7) in eq.(2.3), integrating through the thickness, and carrying out the variation, the following equation is obtained :

$$\int_0^a \int_0^b \Big[\left(u_{o,x} + 1/2.(w_{,x})^2 - \frac{N_x}{Eh} + \frac{\nu N_y}{Eh}\right) \delta N_x$$

$$+ \left(v_{o,y} + 1/2.(w_{,y})^2 - \frac{N_y}{Eh} + \frac{\nu N_x}{Eh}\right) \delta N_y$$

$$+ \left(u_{o,y} + v_{o,x} - \frac{N_{xy}}{Gh} + w_{,x}.w_{,y}\right) \delta N_{xy}$$

$$+ \left(\alpha_{,x} - \frac{12.M_x}{Eh^3} + \frac{12.\nu.M_y}{Eh^3}\right) \delta M_x$$

$$+ \left(\beta_{,y} - \frac{12.M_y}{Eh^3} + \frac{12.\nu.M_x}{Eh^3}\right) \delta M_y$$

$$+ \left(\alpha_{,y} + \beta_{,x} - \frac{12.M_{xy}}{Gh^3}\right) \delta M_{xy}$$

$$+ \left(\alpha + w_{,x} - \frac{6.Q_x}{5Gh}\right) \delta Q_x$$

$$+ \left(\beta + w_{,y} - \frac{6.Q_y}{5Gh}\right) \delta Q_y$$

$$+ \left(-M_{x,x} - M_{xy,y} + Q_x + \frac{\rho . h^3}{12} \ddot{\alpha}\right) \delta\alpha$$

$$+ \left(-M_{y,y} - M_{xy,x} + Q_y + \frac{\rho . h^3 . \ddot{\beta}}{12}\right) \delta\beta$$

$$+ (-N_{x,x} - N_{xy,y} + \rho . h . \ddot{u}_o) \delta u_o$$

$$+ (-N_{y,y} - N_{xy,x} + \rho . h . \ddot{v}_o) \delta v_o$$

$$+ (-N_x . w_{,xx} - N_{x,x} . w_{,x} - N_y . w_{,yy} - N_{y,y} . w_{,y}$$

$$- Q_{x,x} - Q_{y,y} - 2N_{xy} . w_{,xy} - N_{xy,x} . w_{,y}$$

$$- N_{xy,y} . w_{,x} + \rho . h . \ddot{w}) \delta w \Big] d_x d_y$$

$$+ \int_0^b \Big[(N_x) . \delta u_o + (N_{xy}) \delta v_o + (N_x . w_{,x} + Q_x$$

$$+ N_{xy} . w_{,y}) \delta w + (M_x) \delta\alpha + (M_{xy}) \delta\beta \Big]_0^a dy$$

$$+ \int_0^a \Big[(N_y) . \delta v_o + (N_{xy}) \delta u_o + (N_y . w_{,y} + Q_y$$

$$+ N_{xy} . w_{,x}) \delta w + (M_y) \delta\beta + (M_{xy}) \delta\alpha \Big]_0^b dx = 0 \qquad (2.8)$$

The forces and couples are derived as :

$$N_x = \frac{Eh}{(1-\nu^2)} \left[u_{o,x} + \nu . v_{o,y} + 1/2.(w_{,x})^2 + \nu/2 (w_{,y})^2 \right] \qquad (2.9.1)$$

$$N_y = \frac{Eh}{(1-\nu^2)} \left[v_{o,y} + \nu u_{o,x} + 1/2 (w_{,y})^2 + \nu/2 (w_{,x})^2 \right. \qquad (2.9.2)$$

$$N_{xy} = Gh (u_{o,y} + v_{o,x} + w_{,x} . w_{,y}) \qquad (2.9.3)$$

$$M_x = D (\alpha_{,x} + \nu \beta_{,y}) \qquad (2.9.4)$$

$$M_y = D\ (\beta_{,y} + \nu . \alpha_{,x}) \tag{2.9.5}$$

$$M_{xy} = \frac{Gh^3}{12}\ (\alpha_{,y} + \beta_{,x}) \tag{2.9.6}$$

$$Q_x = \frac{5Gh}{6}\ (\alpha + w_{,x}) \tag{2.9.7}$$

$$Q_y = \frac{5Gh}{6}\ (\beta + w_{,y}) \tag{2.9.8}$$

Also, from eq.(2.8) the following equations of motion in terms of components of rotation and displacement are derived :

$$\int_0^a \int_0^b \left\{ \left[-D\ \alpha_{,xx} - \frac{D(1-\nu)}{2} \alpha_{,yy} - \frac{D(1+\nu)}{2} \beta_{,xy} + \frac{5Gh}{6} (\alpha + w_{,x}) + \frac{\rho h^3 \ddot{\alpha}}{12} \right] \delta\alpha \right\} dx\ dy = 0 \tag{2.10.1}$$

$$\int_0^a \int_0^b \left\{ \left[-D\ \beta_{,yy} - \frac{D(1-\nu)}{2} \beta_{,xx} - \frac{D(1+\nu)}{2} \alpha_{,xy} + \frac{5Gh}{6} (\beta + w_{,y}) + \frac{\rho h^3 \ddot{\beta}}{12} \right] \delta\beta \right\} dx\ dy = 0 \tag{2.10.2}$$

$$\int_0^a \int_0^b \left\{ \left[- \frac{Eh}{(1-\nu^2)} u_{o,xx} - \frac{Eh}{2(1-\nu)} v_{o,xy} - \frac{Eh}{(1-\nu^2)} . w_{,x} . w_{,xx} - \frac{Eh}{2(1-\nu)} w_{,y} . w_{,xy} - \frac{Eh}{2(1+\nu)} . u_{o,yy} - \frac{Eh}{2(1+\nu)} . w_{,x} . w_{,yy} + \rho h\ \ddot{u}_o \right] \delta u_o \right\} dx\ dy = 0 \tag{2.10.3}$$

$$\int_0^a \int_0^b \Big\{ \Big[-\frac{Eh}{(1-\nu^2)} v_{o,yy} - \frac{Eh}{2(1-\nu)} u_{o,xy} - \frac{Eh}{(1-\nu^2)} \cdot w_{,y}$$

$$\cdot w_{,yy} - \frac{Eh}{2(1-\nu)} w_{,x} \cdot w_{,xy} - \frac{Eh}{2(1+\nu)} \cdot v_{o,xx} - \frac{Eh}{2(1+\nu)}$$

$$w_{,y} \cdot w_{,xx} + \rho h \ddot{v}_o \Big] \delta v_o \Big\} dx\, dy = 0 \qquad (2.10.4)$$

$$\int_0^a \int_0^b \Big\{ \Big[-\frac{Eh}{(1-\nu^2)} (u_{o,x} \cdot w_{,xx} + v_{o,y} \cdot w_{,yy}) - \frac{\nu Eh}{(1-\nu^2)}$$

$$\cdot (u_{o,x} \cdot w_{,yy} + v_{o,y} \cdot w_{,xx}) - \frac{Eh}{(1+\nu)} (u_{o,y} + v_{o,x}) \cdot w_{,xy}$$

$$- \frac{Eh}{(1-\nu^2)} (u_{o,xx} \cdot w_{,x} + v_{o,yy} \cdot w_{,y}) - \frac{Eh}{2(1-\nu)} (u_{o,xy} \cdot w_{,y}$$

$$+ v_{o,xy} \cdot w_{,x}) - \frac{Eh}{2(1+\nu)} (v_{o,xx} \cdot w_{,y} + u_{o,yy} \cdot w_{,x}) - \frac{5Gh}{6}$$

$$\cdot (\alpha_{,x} + \beta_{,y}) - \frac{3Eh}{2(1-\nu^2)} (w_{,xx} \cdot (w_{,x})^2 + w_{,yy} \cdot (w_{,y})^2)$$

$$- \frac{Eh}{2(1-\nu^2)} (w_{,xx} \cdot (w_{,y})^2 + w_{,yy} \cdot (w_{,x})^2) - \frac{2Eh}{(1-\nu^2)} \cdot w_{,x} \cdot w_{,y} \cdot$$

$$\cdot w_{,xy} - \frac{5Gh}{6} \nabla^2 w + \rho h \ddot{w} \Big] \delta w \Big\} dx\, dy = 0 \qquad (2.10.5)$$

Further, the boundary conditions as given by eq.(2.8) are:

(i) along $x = 0$ and $x = a$

either	or
$\alpha = 0$	$M_x = 0$
$\beta = 0$	$M_{xy} = 0$
$u_o = 0$	$N_x = 0$
$v_o = 0$	$N_{xy} = 0$
$w = 0$	$Q_x + N_x \cdot w_{,x} + N_{xy} \cdot w_{,y} = 0$

(2.11)

(ii) Similarly along $y = 0$ and $y = b$ (2.12).

Equations (2.10) are in the proper form for the application of Galerkin technique. For a plate hinged along all the four edges and vibrating in the first mode, the following approximate expressions for rotation and displacement components can be taken :

$$\alpha = A_1(t) \cdot \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$\beta = B_1(t) \sin \frac{\pi x}{a} \cdot \cos \frac{\pi y}{b}$$

$$u_o = M(t)\, h \cdot \sin \frac{2\pi x}{a} \cdot \cos \frac{2\pi y}{b} \qquad (2.13)$$

$$v_o = N(t)\, h \cdot \cos \frac{2\pi x}{a} \cdot \sin \frac{2\pi y}{b}$$

$$w = P(t)\, h \cdot \sin \frac{\pi x}{a} \cdot \sin \frac{\pi y}{b}$$

These satisfy all the boundary conditions for a hinged plate, namely :

(i) along x = 0 and x = a

$$M_x = 0$$
$$\beta = 0$$
$$u_o = 0$$
$$N_{xy} = 0$$
$$w = 0$$

(ii) similarly along y = 0 and y = b.

Then, from eqs. (2.10), the following equations are derived after integration* :

$$A_1'' + a_1 A_1 + a_2 B_1 + a_3 P = 0 \qquad (2.14.1)$$

$$B_1'' + b_1 B_1 + b_2 A_1 + b_3 P = 0 \qquad (2.14.2)$$

$$M'' + c_1 M + c_2 N + c_3 P^2 = 0 \qquad (2.14.3)$$

$$N'' + d_1 N + d_2 M + d_3 P^2 = 0 \qquad (2.14.4)$$

$$P'' + e_1 P^3 + e_2 P + e_3 PM + e_4 PN + e_5 A_1 + e_6 B_1 = 0 \qquad (2.14.5)$$

---

* From here onwards $E/2(1+\nu)$ is substituted for G in this section.

where,

$$a_1 = \frac{\pi^2}{(1-\nu^2)} + \frac{\pi^2}{2(1+\nu)B^2} + \frac{5}{(1+\nu)H^2}$$

$$a_2 = \frac{\pi^2}{2(1-\nu)\ B}$$

$$a_3 = \frac{5\pi}{(1+\nu)H}$$

$$b_1 = \frac{\pi^2}{(1-\nu^2)B^2} + \frac{\pi^2}{2(1+\nu)} + \frac{5}{(1+\nu)\ H^2}$$

$$b_2 = \frac{\pi^2}{2(1-\nu)B}$$

$$b_3 = \frac{5\pi}{(1+\nu)B.H}$$

$$c_1 = \frac{4\pi^2}{(1-\nu^2)} + \frac{2\pi^2}{(1+\nu)\ B^2}$$

$$c_2 = \frac{2\pi^2}{(1-\nu)B}$$

$$c_3 = -\frac{\pi^3 .\ H}{4(1-\nu^2)}\ (1+\frac{1}{B^2})$$

$$d_1 = \frac{4\pi^2}{(1-\nu^2)B^2} + \frac{2\pi^2}{(1+\nu)} \tag{2.15}$$

$$d_2 = \frac{2\pi^2}{(1-\nu)B}$$

$$d_3 = -\frac{\pi^3 H}{4(1-\nu^2)} \left(\frac{1}{B} + \frac{1}{B^3}\right)$$

$$e_1 = \frac{\pi^4 . H^2}{32(1-\nu^2)} \left(9 + \frac{2}{B^2} + \frac{9}{B^4}\right)$$

$$e_2 = \frac{5\pi^2}{12(1+\nu)} \left(1 + \frac{1}{B^2}\right)$$

$$e_3 = -\frac{\pi^3 . H}{2(1-\nu^2)} \left(1 + \frac{1}{B^2}\right)$$

$$e_4 = -\frac{\pi^3 H}{2(1-\nu^2)B} \left(1 + \frac{1}{B^2}\right)$$

$$e_5 = \frac{5\pi}{12(1+\nu)H}$$

$$e_6 = \frac{5\pi}{12(1+\nu)B.H}$$

In predominantly flexural vibration, in-plane inertia can be neglected (see Ref. 56) and therefore, M" and N" are omitted from eqs.(2.14.3) and (2.14.4). Then all the five eqs.(2.14) are combined into one higher order equation with P as the only dependent variable. Neglecting fourth and higher derivatives of P, which have negligible effect on the fundamental period (see Refs. 2 and 3):

$$g_1 P'' P^2 + g_2 P'' + g_3 (P')^2 . P + g_4 P^3 + g_5 P = 0 \quad (2.16)$$

where,

$$g_1 = (a_1 + b_1)(3e_1 + 3e_3 j_1 + 3e_4 j_2)$$

$$g_2 = (a_1 b_1 - a_2 b_2) + (a_1 + b_1) e_2 - (a_3 e_5 + b_3 e_6)$$

$$g_3 = 6(a_1 + b_1)(e_1 + e_3 j_1 + e_4 j_2)$$

$$g_4 = (a_1 b_1 - a_2 b_2)(e_1 + e_3 j_1 + e_4 j_2) \tag{2.17}$$

$$g_5 = (a_1 b_1 - a_2 b_2) e_2 + (a_2 b_3 - a_3 b_1) e_5 + (a_3 b_2 - a_1 b_3) e_6$$

$$j_1 = \frac{c_2 d_3 - c_3 d_1}{c_1 d_1 - c_2 d_2}$$

$$j_2 = \frac{c_3 d_2 - c_1 d_3}{c_1 d_1 - c_2 d_2}$$

For small amplitude oscillations, terms containing products of P and its derivatives can be neglected giving,

$$g_2 P'' + g_5 P = 0 \tag{2.18}$$

Corresponding non-dimensional period of flexural vibration is obtained as

$$T_L = \frac{2\pi}{\sqrt{g_5/g_2}} \tag{2.19}$$

This is the period of vibration when effects of transverse shear and rotatory inertia are included.

An erroneous but frequently employed application of Galerkin technique is demonstrated below. It is not an uncommon practice to combine the equations derived after variation in some convenient form and then apply Galerkin technique. As for example, let there be n equations derived by the use of variational principle :

$$\iint \left[f_1 \, (F_1, \ldots\ldots, F_n) \, \delta F_1 \right] dx \, dy = 0$$

$$\text{— — — — — —} \tag{2.20}$$

$$\iint \left[f_n \, (F_1, \ldots\ldots, F_n) \, \delta F_n \right] dx \, dy = 0$$

Approximate forms of $F_1, \ldots\ldots, F_n$ can be assumed and the integration can be carried out. This is the correct Galerkin procedure. But if

$$f_1 \, (F_1, \ \ldots\ldots, \ F_n) = 0$$

$$\text{— — — — —} \tag{2.21}$$

$$f_n \, (F_1, \ \ldots\ldots, \ F_n) = 0$$

are combined in such a way that the resulting equations are not coefficient equations of $\delta(\ )$, it is not proper to apply Galerkin technique in this form. This can be shown by carrying out numerical computations.

From eq.(2.8), the governing equations can also be written in the following form :

$$u_{o,x} + 1/2(w_{,x})^2 - \frac{N_x}{Eh} + \frac{\nu N_y}{Eh} = 0 \qquad (2.22.1)$$

$$v_{o,y} + 1/2(w_{,y})^2 - \frac{N_y}{Eh} + \frac{\nu N_x}{Eh} = 0 \qquad (2.22.2)$$

$$u_{o,y} + v_{o,x} - \frac{N_{xy}}{Gh} + w_{,x} \cdot w_{,y} = 0 \qquad (2.22.3)$$

$$\alpha_{,x} - \frac{12M_x}{Eh^3} + \frac{12\nu M_y}{Eh^3} = 0 \qquad (2.22.4)$$

$$\beta_{,y} - \frac{12M_y}{Eh^3} + \frac{12\nu M_x}{Eh^3} = 0 \qquad (2.22.5)$$

$$\alpha_{,y} + \beta_{,x} - \frac{12M_{xy}}{Gh^3} = 0 \qquad (2.22.6)$$

$$\alpha + w_{,x} - \frac{6Q_x}{5Gh} = 0 \qquad (2.22.7)$$

$$\beta + w_{,y} - \frac{6Q_y}{5Gh} = 0 \qquad (2.22.8)$$

$$M_{x,x} + M_{xy,y} - Q_x - \frac{\rho h^3}{12}\ddot{\alpha} = 0 \qquad (2.22.9)$$

$$M_{y,y} + M_{xy,x} - Q_y - \frac{\rho h^3}{12}\ddot{\beta} = 0 \qquad (2.22.10)$$

$$N_{x,x} + N_{xy,y} - \rho h\,\ddot{u}_o = 0 \qquad (2.22.11)$$

$$N_{y,y} + N_{xy,x} - \rho h\,\ddot{v}_o = 0 \qquad (2.22.12)$$

$$N_x \cdot w_{,xx} + N_y \cdot w_{,yy} + 2N_{xy} \cdot w_{,xy} + N_{x,x} \cdot w_{,x} + N_{xy,x} \cdot w_{,y}$$

$$+ N_{xy,y} \cdot w_{,x} + N_{y,y} \cdot w_{,y} + Q_{x,x} + Q_{y,y} - \rho h \ddot{w} = 0 \quad (2.22.13)$$

As done in the previous case, in-plane inertia terms from eqs.(2.22.11) and (2.22.12) can be neglected. Selecting a function $\emptyset$ such that :

$$N_x = \emptyset_{,yy}$$

$$N_y = \emptyset_{,xx} \quad (2.23)$$

$$N_{xy} = - \emptyset_{,xy}$$

these two equations are identically satisfied. Then from eqs. (2.22.1) through (2.22.3), the compatibility equation for middle plane strains can be written as (see Ref. 57):

$$\nabla^4 \emptyset + Eh \left(w_{,xx} \cdot w_{,yy} - (w_{,xy})^2\right) = 0 \quad (2.24)$$

From Eqs. (2.22.4) through (2.22.8) M_x, M_y, M_{xy}, Q_x, and Q_y are derived as given by eqs. (2.9.4) through (2.9.8). Substituting these in eqs. (2.22.9) and (2.22.10)

$$D \cdot \alpha_{,xx} + \frac{D(1-\nu)}{2} \alpha_{,yy} + \frac{D(1+\nu)}{2} \cdot \beta_{,xy}$$

$$- \frac{5Gh}{6} (\alpha + w_{,x}) - \frac{\rho h^3 \ddot{\alpha}}{12} = 0 \quad (2.25)$$

$$D\,\beta_{,yy} + \frac{D(1-\nu)}{2}\cdot\beta_{,xx} + \frac{D(1+\nu)}{2}\cdot\alpha_{,xy}$$

$$-\frac{5Gh}{6}(\beta + w_{,y}) - \frac{\rho h^3 \ddot{\beta}}{12} = 0 \qquad (2.26)$$

Equation (2.22.13) takes the form :

$$\phi_{,xx}\cdot w_{,yy} + \phi_{,yy}\cdot w_{,xx} - 2\,\phi_{,xy}\cdot w_{,xy} + \frac{5Gh}{6}$$

$$(\alpha_{,x} + \beta_{,y} + \nabla^2 w) - \rho h \ddot{w} = 0 \qquad (2.27)$$

Eliminating α and β from eqs. (2.25), (2.26), and (2.27):

$$\left[\frac{D^2(1-\nu)\nabla^4}{2} - \frac{D\rho h^3(3-\nu)\nabla^2}{24}\cdot\frac{\partial^2}{\partial t^2} - \frac{5DEh(3-\nu)\nabla^2}{24(1+\nu)}\right.$$

$$\left. + \frac{25E^2h^2}{144(1+\nu)^2} + \frac{5\rho Eh^4}{72(1+\nu)}\frac{\partial^2}{\partial t^2}\right]\left[\phi_{,xx}\cdot w_{,yy} + \phi_{,yy}\cdot w_{,xx}\right.$$

$$\left. - 2\phi_{,xy}\cdot w_{,xy} + \frac{5Eh}{12(1+\nu)}\nabla^2 w - \rho h \ddot{w}\right] + \frac{5Eh}{12(1+\nu)}\cdot$$

$$\left[\frac{5DEh}{24}\frac{(1-\nu)}{(1+\nu)}\nabla^4 - \frac{25E^2h^2}{144(1+\nu)^2}\nabla^2\right.$$

$$\left. - \frac{5\rho Eh^4}{144(1+\nu)}\nabla^2\cdot\frac{\partial^2}{\partial t^2}\right] w = 0 \qquad (2.28)$$

Approximate forms of w and ϕ are selected such that eq.(2.24) is identically satisfied. Equation (2.28) is satisfied on the average by the application of Galerkin technique. It may however, be mentioned that all the five

boundary conditions cannot be satisfied since only two functions are required to be selected in this reduced form of equations. Let the following forms be assumed for w and ϕ for a plate hinged along the edges :

$$w = f(t).h \operatorname{Sin} \frac{\pi x}{a} \cdot \operatorname{Sin} \frac{\pi y}{a} \tag{2.29}$$

$$\phi = f^2(t) \frac{Eh^3}{32} \left[\frac{2\pi^2}{a^2(1-\nu)} (x^2 + y^2) + \operatorname{Cos} \frac{2\pi x}{a} + \operatorname{Cos} \frac{2\pi y}{a} \right] \tag{2.30}$$

This part of the calculation is limited to a square plate of sides a. The boundary conditions satisfied by these functions are :

(i) along $x = 0$ and $x = a$,

$$\begin{aligned} u_o &= 0 \\ N_{xy} &= 0 \\ w &= 0 \\ M_x &\doteq 0 \end{aligned} \tag{2.31}$$

(ii) similarly along $y = 0$ and $y = a$ (2.32)

The boundary condition on β along $x = 0$ and $x = a$ however, cannot be satisfied.

Substituting these expressions for w and $\emptyset$ in eq.(2.28) and multiplying by Sin $\frac{\pi x}{a}$ Sin $\frac{\pi y}{a}$, integrating over the area of the plate, the following equation is obtained after suitable non-dimensionalisation :

$$a_1 \; f'' \; f^2 + a_2 \; f'' + a_3 (f')^2 f + a_4 \; f + a_5 \; f^3 = 0 \qquad (2.33)$$

where,

$$a_1 = \frac{\pi^6 (3-\nu)^2 H^6}{1536(1-\nu^2)(1-\nu)} + \frac{5\pi^4 (3-\nu).H^4}{768(1-\nu^2)}$$

$$a_2 = \frac{\pi^4 (27-5\nu) H^4}{3456(1-\nu^2)(1+\nu)} + \frac{5\pi^2 (23-11\nu).H^2}{3456(1-\nu^2)(1+\nu)} + \frac{25}{576(1+\nu)^2}$$

$$a_3 = \frac{\pi^6 (3-\nu)^2 H^6}{768(1-\nu^2)(1-\nu)} + \frac{5\pi^4 (3-\nu) H^4}{384(1-\nu^2)} \qquad (2.34)$$

$$a_4 = \frac{5\pi^6 \; H^4}{1728(1-\nu^2)(1+\nu)^2} + \frac{25\pi^4 \; H^2}{1728(1-\nu^2)(1+\nu)^2}$$

$$a_5 = \frac{\pi^8 (3-\nu) \; H^6}{2304(1-\nu^2)^2} + \frac{5\pi^6 (3-\nu)^2 H^4}{4608(1-\nu^2)^2} + \frac{25\pi^4 (3-\nu) \; H^2}{4608(1+\nu)^2 (1-\nu)}$$

The fourth and higher derivatives of time have been neglected in deriving eq.(2.33) as has already been explained earlier.

2.2 THE CASE OF A STRAIGHT BEAM

From the equations of a rectangular plate, the equations for a beam (see Fig.3) can be specialised by taking variables connected with y direction and the Poisson's ratio ν to be zero*.

The stress resultants derived from eqs.(2.9) are :

$$\begin{aligned} N_x &= Eh\,(u_{o,x} + 1/2\,(w_{,x})^2) \\ M_x &= \frac{Eh^3}{12}\,\alpha_{,x} \\ Q_x &= \frac{5Gh}{6}\,(\alpha + w_{,x}) \end{aligned} \tag{2.35}$$

The variational equations as derived from eqs.(2.10) are :

$$\int_0^a \left[\left(- \frac{Eh^3}{12}\,\alpha_{,xx} + \frac{5Gh}{6}\,(\alpha + w_{,x}) + \frac{\rho h^3 \ddot{\alpha}}{12}\right)\delta\alpha \right] dx = 0 \tag{2.36.1}$$

$$\int_0^a \left[(-Eh\, u_{o,xx} - Eh\, w_{,x} \cdot w_{,xx})\, \delta u_o \right] dx = 0 \tag{2.36.2}$$ **

---

* This is to be done before $\frac{E}{2(1+\nu)}$ is substituted for G.

** In-plane inertia term has been neglected.

$$\int_0^a \Big[\Big(-Eh\, u_{o,x} \cdot w_{,xx} - Eh\, u_{o,xx} \cdot w_{,x} - \frac{5Gh}{6} \alpha_{,x} \cdot$$

$$- \frac{3}{2} Eh \cdot w_{,xx} \cdot (w_{,x})^2 - \frac{5Gh}{6} w_{,xx} + \rho h \ddot{w} \Big) \delta w \Big] dx = 0 \tag{2.36.3}$$

The boundary conditions along $x = 0$ and $x = a$ specialised from eqs. (2.11) are :

either	or
$\alpha = 0$	$M_x = 0$
$u_o = 0$	$N_x = 0$
$w = 0$	$Q_x + N_x \cdot w_{,x} = 0$

(2.37)

For a beam hinged at both the ends and vibrating in the fundamental mode, the following forms for α, u_o, and w can be taken :

$$\alpha = A_1 (t) \operatorname{Cos} \frac{\pi x}{a}$$

$$u_o = M (t) . h \operatorname{Sin} \frac{2\pi x}{a} \tag{2.38}$$

$$w = P (t) . h \operatorname{Sin} \frac{\pi x}{a}$$

Then from eqs. (2.36) the following eqs. are derived :

$$A_1'' + a_1 A_1 + a_2 P = 0$$

$$M'' + b_1 M + b_2 P^2 = 0 \tag{2.39}$$

$$P'' + c_1 P^3 + c_2 P + c_3 A + c_4 PM = 0$$

where,

$$a_1 = \pi^2 + \frac{5}{(1+\nu)H^2}$$

$$a_2 = \frac{5\pi}{(1+\nu)H}$$

$$b_1 = 4\pi^2$$

$$b_2 = -\frac{\pi^3 H}{4} \tag{2.40}$$

$$c_1 = \frac{9\pi^4 H^2}{32}$$

$$c_2 = \frac{5\pi^2}{12(1+\nu)}$$

$$c_3 = \frac{5\pi}{12H(1+\nu)}$$

$$c_4 = -\frac{\pi^3 H}{2}$$

As in the previous case, in-plane inertia is neglected and the three eqs. (2.39) are combined into one equation with P as the only dependent variable. Retaining terms containing derivatives of P only upto the second order :

$$g_1 P'' P^2 + g_2 P'' + g_3 P(P')^2 + g_4 P^3 + g_5 P = 0 \tag{2.41}$$

where,

$$g_1 = \frac{3}{4}\pi^4 H^2$$

$$g_2 = \pi^2 + \frac{5}{(1+\nu)H^2} + \frac{5\pi^2}{12(1+\nu)}$$

$$g_3 = \frac{3}{2}\pi^4 H^2 \qquad (2.42)$$

$$g_4 = \frac{\pi^6 H^2}{4} + \frac{5\pi^4}{4(1+\nu)}$$

$$g_5 = \frac{5\pi^4}{12(1+\nu)}$$

For small amplitude oscillations, terms containing products of P and its derivatives can be neglected giving,

$$g_2 P'' + g_5 P = 0 \qquad (2.43)$$

Corresponding non-dimensional period of flexural vibration,

$$T_L = \frac{2\pi}{\sqrt{g_5/g_2}} \qquad (2.44)$$

2.3 THE CASE OF A DOUBLY CURVED SHELL ON RECTANGULAR PLAN

A shallow shell (see Fig.4), whose middle surface is given by $W(x,y) = c \operatorname{Sin} \frac{\pi x}{a} . \operatorname{Sin} \frac{\pi y}{b}$, is considered here. c is the maximum initial rise in unstressed condition. For a shell to be shallow c/b should not be greater than $\frac{1}{5}$. For such a shell, the first fundamental form of the surface can be taken as $ds^2 = dx^2 + dy^2$ and any integration required to be carried over the surface of the shell can be carried over the projected plan area without much error (Refs. 58, 59). This means that projected forces and moments can be taken as the actual forces and moments. These are the usual simplifications done in the case of shallow shells.

The components of displacement and distribution of stresses through the thickness are assumed as those given by eqs. (2.5) and (2.7).

The components of nonlinear strains are taken on the basis of the derivation given by Marguerre[60] for shallow shells, namely,

$$\begin{aligned} \epsilon_{xx} &= u_{,x} + 1/2\,(w_{,x})^2 + W_{,x} . w_{,x} \\ \epsilon_{yy} &= v_{,y} + 1/2\,(w_{,y})^2 + W_{,y} . w_{,y} \\ \epsilon_{zz} &\doteq 0 \end{aligned} \tag{2.45}$$

$$\gamma_{xy} = u_{,y} + v_{,x} + w_{,x} \cdot w_{,y} + W_{,x} \cdot w_{,y} + W_{,y} \cdot w_{,x}$$

$$\gamma_{xz} = \alpha + w_{,x}$$

$$\gamma_{yz} = \beta + w_{,y}$$

Substituting for u, v, and w :

$$\epsilon_{xx} = u_{o,x} + z \cdot \alpha_{,x} + 1/2\,(w_{,x})^2 + \frac{c\,\pi}{a} \cdot w_{,x} \cdot \cos\frac{\pi x}{a} \cdot \sin\frac{\pi y}{b}$$

$$\epsilon_{yy} = v_{o,y} + z \cdot \beta_{,y} + 1/2\,(w_{,y})^2 + \frac{c\,\pi}{b}\, w_{,y} \cdot \cos\frac{\pi y}{b} \cdot \sin\frac{\pi x}{a}$$

$$\epsilon_{zz} \doteq 0$$

$$\gamma_{xy} = u_{o,y} + v_{o,x} + z \cdot (\alpha_{,y} + \beta_{,x}) + w_{,x} \cdot w_{,y} + \frac{c\,\pi}{a}\, w_{,y} \cos\frac{\pi x}{a} \cdot \sin\frac{\pi y}{b} + \frac{c\,\pi}{b}\, w_{,x} \cdot \sin\frac{\pi x}{a} \cdot \cos\frac{\pi y}{b}$$

$$\gamma_{xz} = \alpha + w_{,x}$$

$$\gamma_{yz} = \beta + w_{,y}$$

By the variational approach, as in the case of the rectangular plate, the force-displacement relations are obtained as :

$$N_x = \frac{Eh}{(1-\nu^2)} \Big[u_{o,x} + \nu\, v_{o,y} + 1/2\, (w_{,x})^2 + \frac{\nu}{2}\, (w_{,y})^2$$

$$+ \frac{c\pi}{a} \cdot w_{,x} \cdot \mathrm{Cos}\, \frac{\pi x}{a} \cdot \mathrm{Sin}\, \frac{\pi y}{b} + \frac{\nu c \pi}{b}\, w_{,y} \cdot$$

$$\mathrm{Sin}\, \frac{\pi x}{a} \cdot \mathrm{Cos}\, \frac{\pi y}{b} \Big]$$

$$N_y = \frac{Eh}{(1-\nu^2)} \Big[v_{o,y} + \nu\, u_{o,x} + 1/2\, (w_{,y})^2 + \frac{\nu}{2}\, (w_{,x})^2$$

$$+ \frac{c\pi}{b}\, w_{,y} \cdot \mathrm{Sin}\, \frac{\pi x}{a} \cdot \mathrm{Cos}\, \frac{\pi y}{b} + \frac{\nu c \pi}{a}\, w_{,x} \cdot$$

$$\mathrm{Cos}\, \frac{\pi x}{a} \cdot \mathrm{Sin}\, \frac{\pi y}{b} \Big]$$

$$N_{xy} = Gh \Big[u_{o,y} + v_{o,x} + w_{,x} \cdot w_{,y} + \frac{c\pi}{a} \cdot w_{,y} \cdot$$

$$\mathrm{Cos}\, \frac{\pi x}{a} \cdot \mathrm{Sin}\, \frac{\pi y}{b} + \frac{c\pi}{b} \cdot w_{,x} \cdot \mathrm{Sin}\, \frac{\pi x}{a} \cdot \mathrm{Cos}\, \frac{\pi y}{b} \Big]$$

$$M_x = D\, (\alpha_{,x} + \nu \cdot \beta_{,y}) \tag{2.47}$$

$$M_y = D\, (\beta_{,y} + \nu \cdot \alpha_{,x})$$

$$M_{xy} = \frac{Gh^3}{12}\, (\alpha_{,y} + \beta_{,x})$$

$$Q_x = \frac{5Gh}{6}\, (\alpha + w_{,x})$$

$$Q_y = \frac{5Gh}{6}\, (\beta + w_{,y})$$

The boundary conditions as derived are :

(i) along $x = 0$ and $x = a$

either	Or
$\alpha = 0$	$M_x = 0$
$\beta = 0$	$M_{xy} = 0$
$u_o = 0$	$N_x = 0$
$v_o = 0$	$N_{xy} = 0$
$w = 0$	$Q_x + N_x \cdot w_{,x} + N_{xy} \cdot w_{,y} + \frac{c\pi}{a} N_x \cdot \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{c\pi}{b} N_{xy} \cdot \sin\frac{\pi x}{a} \cdot \cos\frac{\pi y}{b} = 0$

(2.48)

(ii) similarly along $y = 0$ and $y = b$ (2.49)

The components of rotation and displacement, for a shell hinged along the four edges, are taken to be the same as in eqs. (2.13) for the fundamental mode of vibration.

Ultimately the governing equations in non-dimensional form (corresponding to eqs.(2.14) for the rectangular plate) reduce to :

$$A_1'' + a_1 A_1 + a_2 B_1 + a_3 P = 0$$

$$B_1'' + b_1 B_1 + b_2 A_1 + b_3 P = 0$$

$$M'' + c_1 M + c_2 N + c_3 P^2 + c_4 P = 0 \qquad (2.50)$$

$$N'' + d_1 N + d_2 M + d_3 P^2 + d_4 P = 0$$

$$P'' + e_1 P^3 + e_2 P^2 + e_3 P + e_4 PM + e_5 PN + e_6 M$$
$$+ e_7 N + e_8 A_1 + e_9 . B_1 = 0$$

where,

$$a_1 = \frac{\pi^2}{(1-\nu^2)} + \frac{\pi^2}{2(1+\nu)B^2} + \frac{5}{(1+\nu)H^2}$$

$$a_2 = \frac{\pi^2}{2(1-\nu)B}$$

$$a_3 = \frac{5\pi}{(1+\nu)\ H}$$

$$b_1 = \frac{\pi^2}{(1-\nu^2)B^2} + \frac{\pi^2}{2(1+\nu)} + \frac{5}{(1+\nu)H^2}$$

$$b_2 = \frac{\pi^2}{2(1-\nu)\ B}$$

$$b_3 = \frac{5\pi}{(1+\nu)B.H}$$

$$
\begin{aligned}
c_1 &= \frac{4\pi^2}{(1-\nu^2)} + \frac{2\pi^2}{(1+\nu)B^2} \\
c_2 &= \frac{2\pi^2}{(1-\nu)B} \\
c_3 &= -\frac{\pi^3 H}{4(1-\nu^2)}\left(1+\frac{1}{B^2}\right) \\
c_4 &= -\frac{k\pi^3 H}{2(1-\nu^2)}\left(1+\frac{1}{B^2}\right) \\
d_1 &= \frac{4\pi^2}{(1-\nu^2)B^2} + \frac{2\pi^2}{(1+\nu)} \\
d_2 &= \frac{2\pi^2}{(1-\nu)B} \\
d_3 &= -\frac{\pi^3 H}{4(1-\nu^2)}\left(\frac{1}{B}+\frac{1}{B^3}\right) \\
d_4 &= -\frac{k\pi^3 H}{2(1-\nu^2)}\left(\frac{1}{B}+\frac{1}{B^3}\right) \\
e_1 &= \frac{\pi^4 H^2}{32(1-\nu^2)}\left(9+\frac{9}{B^4}+\frac{2}{B^2}\right) \\
e_2 &= \frac{3\pi^4 H^2 k}{32(1-\nu^2)}\left(9+\frac{2}{B^2}+\frac{9}{B^4}\right)
\end{aligned}
\tag{2.51}
$$

$$e_3 = \frac{k^2 \pi^4 H^2}{(1-\nu^2)} \left(\frac{9}{16} + \frac{9}{16B^4} + \frac{1}{8B^2}\right) + \frac{5\pi^2}{12(1+\nu)}\left(1+\frac{1}{B^2}\right)$$

$$e_4 = -\frac{\pi^3 H}{2(1-\nu^2)}\left(1+\frac{1}{B^2}\right)$$

$$e_5 = -\frac{\pi^3 H}{2(1-\nu^2)B}\left(1+\frac{1}{B^2}\right)$$

$$e_6 = -\frac{k\pi^3 H}{2(1-\nu^2)}\left(1+\frac{1}{B^2}\right)$$

$$e_7 = -\frac{k\pi^3 H}{2(1-\nu^2)B}\left(1+\frac{1}{B^2}\right)$$

$$e_8 = \frac{5}{12(1+\nu)H}$$

$$e_9 = \frac{5\pi}{12(1+\nu)B.H}$$

Neglecting in-plane inertia terms in eqs.(2.50), combining these in a way similar to that in the case of the rectangular plate, and neglecting terms containing fourth and higher order derivatives with respect to time, the following equation is obtained :

$$g_1P''P^2+g_2P''P+g_3P''+g_4P(P')^2+g_5(P')^2+g_6P^3+g_7P^2+g_8P = 0 \qquad (2.52)^*$$

---

*Comparing eq.(2.16) with eq.(2.52), it is seen that quadratic nonlinearity is not present in the former. Its implication will be discussed in Chapter III.

'here,

$$g_1 = 3(a_1 + b_1) \cdot (e_1 + e_4 j_1 + e_5 j_3)$$

$$g_2 = 2(a_1 + b_1) \cdot (e_2 + e_4 j_2 + e_5 j_4 + e_6 j_1 + e_7 j_3)$$

$$g_3 = a_1 b_1 - a_2 b_2 + (a_1 + b_1) \cdot (e_3 + e_6 j_2 + e_7 j_4) - a_3 e_8 - b_3 e_9$$

$$g_4 = 6(a_1 + b_1)(e_1 + e_4 j_1 + e_5 j_3)$$

$$g_5 = 2(a_1 + b_1)(e_2 + e_4 j_2 + e_5 j_4 + e_6 j_1 + e_7 j_3)$$

$$g_6 = (a_1 b_1 - a_2 b_2)(e_1 + e_4 j_1 + e_5 j_3) \tag{2.53}$$

$$g_7 = (a_1 b_1 - a_2 b_2)(e_2 + e_4 j_2 + e_5 j_4 + e_6 j_1 + e_7 j_3)$$

$$g_8 = (a_1 b_1 - a_2 b_2)(e_3 + e_6 j_2 + e_7 j_4) + e_8(a_2 b_3 - a_3 b_1) + e_9(a_3 b_2 - a_1 b_3)$$

$$j_1 = \frac{c_2 d_3 - c_3 d_1}{c_1 d_1 - c_2 d_2}$$

$$j_2 = \frac{c_2 d_4 - c_4 d_1}{c_1 d_1 - c_2 d_2}$$

$$j_3 = \frac{c_3 d_2 - c_1 d_3}{c_1 d_1 - c_2 d_2}$$

$$j_4 = \frac{c_4 d_2 - c_1 d_4}{c_1 d_1 - c_2 d_2}$$

For small amplitude oscillations, terms containing products of P and its derivatives are neglected in eq.(2.52), giving,

$$g_3 P'' + g_8 P = 0 \qquad (2.54)$$

Corresponding non-dimensional period of flexural vibration is

$$T_L = \frac{2\pi}{\sqrt{g_8/g_3}} \qquad (2.55)$$

It can be seen that equations for the case of a rectangular plate derived in section 2.1 are obtained by putting c equal to zero in the corresponding equations of the present section.

2.4 THE CASE OF A SHALLOW ARCH

A shallow arch (see Fig.5) whose middle surface is given by $W = c \operatorname{Sin} \frac{\pi x}{a}$ is considered here. c gives the initial maximum rise in unstressed condition. For the arch to be shallow c/a should not be greater than 1/5. All the equations for this case can be derived as a special case by omitting variables connected with y direction and putting ν equal to zero in the corresponding equations for the doubly curved shell given in the previous section*.

---

* This has to be done before $\frac{E}{2(1+\nu)}$ is substituted for G.

The stress resultants are :

$$N_x = Eh\left(u_{o,x} + 1/2\,(w_{,x})^2 + \frac{c\,\pi}{a}\, w_{,x} \cos\frac{\pi x}{a}\right)$$

$$M_x = \frac{Eh^3}{12}\ \alpha_{,x} \qquad (2.56)$$

$$Q_x = \frac{5Gh}{6}\,(\alpha + w_{,x})$$

The boundary conditions along $x = 0$ and $x = a$ are :

<u>either</u>	<u>or</u>	
$\alpha = 0$	$M_x = 0$	
$u_o = 0$	$N_x = 0$	(2.57)
$w = 0$	$Q_x + N_x.w_{,x} + \frac{c\,\pi}{a}\ .\ N_x.\cos\frac{\pi x}{a} = 0$	

For an arch hinged at both the ends and vibrating in the fundamental mode, rotation and displacements are assumed as,

$$\alpha = A_1(t)\cos\frac{\pi x}{a}$$

$$u_o = M(t)h\sin\frac{2\pi x}{a} \qquad (2.58)$$

$$w = P(t)\,h\sin\frac{\pi x}{a}$$

Ultimately the second order equation, corresponding to eq.(2.52) of the previous section, comes out to be :

$$g_1 P'' P^2 + g_2 P'' P + g_3 P'' + g_4 P (P')^2 + g_5 (P')^2 + g_6 P^3$$

$$+ g_7 P^2 + g_8 P = 0 \qquad (2.59)^*$$

where,

$$g_1 = \frac{3}{4} \pi^4 H^2$$

$$g_2 = \frac{3}{2} \cdot k \cdot \pi^4 H^2$$

$$g_3 = \pi^2 + \frac{5}{(1+\nu) H^2} + \frac{5 \pi^2}{12(1+\nu)} + \frac{1}{2} k^2 \pi^4 . H^2$$

$$g_4 = 2 g_1$$

$$g_5 = g_2 \qquad (2.60)$$

$$g_6 = \frac{\pi^6 H^2}{4} + \frac{5 \pi^4}{4(1+\nu)}$$

$$g_7 = \frac{3}{4} \pi^6 H^2 k + \frac{15}{4} \frac{\pi^4 k}{(1+\nu)}$$

$$g_8 = \frac{5 \pi^4}{12(1+\nu)} + \frac{k^2 \pi^4 H^2}{2} \left(\pi^2 + \frac{5}{(1+\nu) H^2} \right)$$

For small amplitude oscillations, terms containing products of P and its derivatives are neglected, giving,

$$g_3 P'' + g_8 P = 0 \qquad (2.61)$$

Corresponding non-dimensional period of linear vibration is

$$T_L = \frac{2\pi}{\sqrt{g_8/g_3}} \qquad (2.62)$$

---

* Comparing eq.(2.41) with eq.(2.59) it is seen that quadratic nonlinearity is not present in the former.

2.5 THE CASE OF A SPHERICAL CAP

Axisymmetric (torsionless) motion of a spherical cap has been analysed in this section. The equations of motion given by Ogibalor[59] for the case of a shallow spherical shell (Fig.6) are taken in this work after modifying them to suit the sign conventions used here. These force and moment equations were obtained by suitably integrating through the thickness, the equations of motion derived from the theory of elasticity. A spherical shell is considered to be shallow if $\frac{a}{H^*} > 4$ (Ref.47). For such a shell, the actual values of forces and moments can be taken to be the projections of these in plan.

The equations of motion are :

$$T_{rr,r} + \frac{T_{rr} - T_{\theta\theta}}{r} + (\emptyset . Q)_{,r} + \frac{\emptyset Q}{r} + \frac{Q}{R} = \rho h \ddot{u}_r \qquad (2.63.1)$$

$$Q_{,r} + \frac{Q}{r} + (T_{rr} . w_{,r})_{,r} + \frac{T_{rr} \cdot w_{,r}}{r} - \frac{1}{R}(T_{rr} + \emptyset Q + T_{\theta\theta}) = \rho h \ddot{w} \qquad (2.63.2)$$

$$M_{rr,r} + \frac{M_{rr} - M_{\theta\theta}}{r} - Q = \frac{\rho h^3}{12} \ddot{\emptyset} \qquad (2.63.3)$$

where $u^z (r,z,t) = u_r(r,t) + z.\emptyset(r,t)$ (2.64)

$$Q = k_s Gh \, (w_{,r} + \emptyset) \qquad (2.65)$$

In eq.(2.65) k_s is a correction factor introduced to take care of distribution of shear stress through the thickness. For the case of a plate, its value has been derived by Mindlin[61] to be $\frac{\pi^2}{12}$ by adjusting the lowest simple thickness-shear frequency of a homogeneous plate to its exact value. This value will be taken in all the computations here.

The force-displacement and moment-curvature relations for the case of a spherical shell are taken as (Ref.62) :

$$
\begin{aligned}
T_{rr} &= \frac{Eh}{(1-\nu^2)}(\epsilon^o_{rr} + \nu\,\epsilon^o_{\theta\theta}) \\
T_{\theta\theta} &= \frac{Eh}{(1-\nu^2)}(\epsilon^o_{\theta\theta} + \nu\,\epsilon^o_{rr}) \\
M_{rr} &= D(k_{rr} + \nu\, k_{\theta\theta}) \\
M_{\theta\theta} &= D(k_{\theta\theta} + \nu\, k_{rr})
\end{aligned}
\tag{2.66}
$$

The strains and curvatures are taken as (Ref.59) :

$$
\begin{aligned}
\epsilon^o_{rr} &= u_{r,r} + \frac{w}{R} + \frac{1}{2}(w_{,r})^2 \\
\epsilon^o_{\theta\theta} &= \frac{u_r}{r} + \frac{w}{R} \\
k_{rr} &= \phi_{,r} \\
k_{\theta\theta} &= \phi/r
\end{aligned}
\tag{2.67}
$$

These are valid for moderately thick shells considered in this work; the transverse deflection is large (upto 2 or 3 times the thickness of the shell), elongations and shears are small but rotations are moderately large. Assuming the shell to be hinged along the periphery, w, $\emptyset$, and u_r are taken in the following form :

$$w = A_1(t)\, h\, S_o(\lambda r)$$

$$\emptyset = B_1(t)\, S_1(\lambda r) \qquad (2.68)$$

$$u_r = C(t)\left(\frac{r^2}{a} - r\right)$$

where,

$$S_o(\lambda r) = J_o(\lambda r) - f\, I_o(\lambda r)$$

$$S_1(\lambda r) = J_1(\lambda r) + f\, I_1(\lambda r) \qquad (2.69)$$

These functions are assumed on the basis of linear classical solution of a circular plate.

The boundary conditions, at $r = a$, for a hinged shell are :

$$w = 0 \qquad (2.70.1)$$

$$M_r = 0 \qquad (2.70.2)$$

$$u_r = 0 \qquad (2.70.3)$$

Boundary condition (2.70.3) is satisfied by the function selected for u_r. The two constants $\lambda . a$ and f

are obtained by substituting the expressions of w and Ø in eqs. (2.70.1) and (2.70.2).

Writing the equations of motion in terms of rotation and displacements, neglecting in-plane inertia, and applying Gaterkin technique, the following equations are obtained :

$$a_1A_1 + a_2B_1 + a_3A_1^2 + a_4A_1B_1 + a_5B_1^2 - C = 0 \qquad (2.71.1)$$

$$b_1A_1 + b_2B_1 + b_3A_1C + b_4A_1^2 + b_5A_1^3 + b_6B_1^2 + b_7C$$
$$+ b_8A_1B_1 + b_9A_1'' = 0 \qquad (2.71.2)$$

$$B_1'' + c_1B_1 - c_2A_1 = 0 \qquad (2.71.3)$$

where,

$$a_1 = \left\{ 2(1+\nu).H + (1-\nu)\, H.k_s \right\}.\, X_1.\, R^*$$

$$a_2 = -\frac{(1-\nu)k_s}{\lambda a} .\, X_1.R^*$$

$$a_3 = 2H^2 \left(X_2 + \frac{(1-\nu)}{2} .\, X_3\right)$$

$$a_4 = -\frac{(1-\nu)k_s.H}{\lambda a} (X_3 + 2.X_2)$$

$$a_5 = \frac{(1-\nu)k_s}{(\lambda a)^2} (X_3 + 2.X_2)$$

$$b_1 = \frac{(1-\nu)k_s}{2} (X_4 + X_5) - 2(1+\nu).X_{16}.(R*)^2$$

$$b_2 = -\frac{(1-\nu)k_s}{2.H.\ \lambda a} (X_4 + X_5)$$

$$b_3 = X_6 + (1+\nu)\ X_{12} + \nu X_9 + 2.X_{10}$$

$$b_4 = (1+\nu).H.R*\ (X_7 + \frac{X_{11}}{2} + X_{13})$$

$$b_5 = \frac{H^2}{2} (3.X_8 + X_{14})$$

$$b_6 = -\frac{(1-\nu)k_s.X_{11}.R*}{2H.(\lambda a)^2}$$

$$b_7 = -\frac{(1+\nu)}{H} .R^*.(X_{15} + X_{17})$$

$$b_8 = \frac{(1-\nu)\ k_s.X_{11}}{2.\ \lambda\ a} .\ R^*$$

$$b_9 = -(1-\nu^2)\ .\ X_{16}$$

$$c_1 = -\frac{1}{(1-\nu^2).X_{18}} (X_{19} + X_{20} - X_{21}) + \frac{6k_s}{(1+\nu)H^2}$$

$$c_2 = \frac{6.\lambda a.\ k_s}{(1+\nu)H}$$

$$
\begin{aligned}
X_1 &= \int_0^a S_{o,r} \left\{ \left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right) \right\} dr \\
X_2 &= \int_0^a S_{o,r} \cdot S_{o,rr} \, (r^2 - ar) \, dr \\
X_3 &= \int_0^a (S_{o,r})^2 \, (r-a) \, dr \\
X_4 &= a \int_0^a S_o \cdot S_{o,rr} \, dr \\
X_5 &= a \int_0^a \frac{S_o \cdot S_{o,r}}{r} \, dr \\
X_6 &= \int_0^a S_o \cdot S_{o,rr} \cdot (2r-a) \, dr \\
X_7 &= a \int_0^a S_o^2 \cdot S_{o,rr} \, dr \\
X_8 &= a^3 \int_0^a S_o \cdot (S_{o,r})^2 \cdot S_{o,rr} \cdot dr \\
X_9 &= \int_0^a S_o \cdot S_{o,rr} \, (r-a) \, dr \\
X_{10} &= \int_0^a S_o \cdot S_{o,r} \, dr
\end{aligned}
\tag{2.72}
$$

$$X_{11} = a \int_0^a S_o \cdot (S_{o,r})^2 \, dr$$

$$X_{12} = \int_0^a S_o \cdot S_{o,r} \left(2 - \frac{a}{r}\right) dr$$

$$X_{13} = a \int_0^a \frac{S_o^2 \cdot S_{o,r}}{r} \, dr$$

$$X_{14} = a^3 \int_0^a \frac{S_o \cdot (S_{o,r})^3 dr}{r}$$

$$X_{15} = \frac{1}{a^2} \int_0^a S_o \cdot (2r-a) dr$$

$$X_{16} = \frac{1}{a} \int_0^a S_o^2 \, dr$$

$$X_{17} = \frac{1}{a^2} \int_0^a S_o \cdot (r-a) dr$$

$$X_{18} = a \int_0^a (S_{o,r})^2 \, dr$$

$$X_{19} = a^3 \int_0^a S_{o,rrr} \cdot S_{o,r} \, dr$$

$$X_{20} = a^3 \int_0^a \frac{S_{o,rr} \cdot S_{o,r}}{r} \, dr$$

$$X_{21} = a^3 \int_0^a \frac{(S_{o,r})^2}{r^2}\, dr$$

It does not seem to be possible to get one equation in A_1, though, however it is possible to obtain one equation in B_1. Combining the three equations (2.71), neglecting derivatives of B_1 of the order higher than the second, the following equation is obtained :

$$d_1B_1 + d_2B_1^2 + d_3B_1^3 + d_4B_1'' + d_5B_1B_1'' + d_6B_1^2B_1'' = 0 \tag{2.73}$$

Making use of eqs.(2.71) and (2.73) :

$$A_1 = \frac{c_1B_1}{c_2} - \frac{d_1B_1+d_2B_1^2+d_3B_1^3}{c_2(d_4+d_5B_1+d_6B_1^2)} \tag{2.74}$$

where,

$$d_1 = \frac{b_1c_1}{c_2} + b_2 + \frac{a_1b_7c_1}{c_2} + a_2b_7$$

$$d_2 = \frac{a_3b_7c_1^2}{c_2^2} + \frac{a_4b_7c_1}{c_2} + a_5b_7 + \frac{b_4c_1^2}{c_2^2} + b_6 + \frac{b_8c_1}{c_2} + \frac{a_1b_3c_1^2}{c_2^2} + \frac{a_2b_3c_1}{c_2}$$

$$d_3 = \frac{a_3b_3c_1^3}{c_2^3} + \frac{a_4b_3c_1^2}{c_2^2} + \frac{a_5b_3c_1}{c_2} + \frac{b_5c_1^3}{c_2^3}$$

(2.75)

$$d_4 = \frac{b_1}{c_2} + \frac{b_9 c_1}{c_2} + \frac{a_1 b_7}{c_2}$$

$$d_5 = \frac{2a_3 b_7 c_1}{c_2^2} + \frac{a_4 b_7}{c_2} + \frac{2b_4 c_1}{c_2^2} + \frac{b_8}{c_2} + \frac{2a_1 b_3 c_1}{c_2^2} + \frac{a_2 b_3}{c_2}$$

$$d_6 = \frac{2a_4 b_3 c_1}{c_2^2} + \frac{3a_3 b_3 c_1^2}{c_2^3} + \frac{a_5 b_3}{c_2} + \frac{3b_5 c_1^2}{c_2^3}$$

Once B_1 is solved from eq.(2.73), A_1 can be obtained from the cubic equation (2.74).

For small amplitude oscillations, terms containing products of B_1 and its derivatives in eq.(2.73) are neglected, giving,

$$d_1 B_1 + d_4 B_1'' = 0 \qquad (2.76)$$

Corresponding period of linear oscillations is :

$$T_L = \frac{2\pi}{\sqrt{d_1/d_4}} \qquad (2.77)$$

2.6 THE CASE OF A CIRCULAR PLATE

The problem of a circular plate can be reduced from that of a spherical cap discussed in the last section. On putting R* to be zero, the final governing equation (2.73) reduces to :

$$d_1 B_1 + d_2 B_1^3 + d_3 B_1'' + d_4 B_1^2 B_1'' = 0 \qquad (2.78)^{+}$$

+ Comparing eq.(2.78) with eq.(2.73), it is seen that quadratic nonlinearity is not present in eq.(2.78).

Where,

$$d_1 = \frac{b_1 c_1}{c_2} + b_2$$

$$d_2 = \frac{a_3 b_3 c_1^3}{c_2^3} + \frac{a_4 b_3 c_1^2}{c_2^2} + \frac{a_5 b_3 c_1}{c_2} + \frac{b_5 c_1^3}{c_2^3}$$

$$d_3 = \frac{b_1}{c_2} + \frac{b_9 c_1}{c_2} \qquad (2.79)$$

$$d_4 = \frac{2a_4 b_3 c_1}{c_2^2} + \frac{3a_3 b_3 c_1^2}{c_2^3} + \frac{a_5 b_3}{c_2} + \frac{3b_5 c_1^2}{c_2^3}$$

a_3,------, b_1,------, c_1,------ are defined as in the the previous section.

The period of linear oscillation is :

$$T_L = \frac{2\pi}{\sqrt{d_1/d_3}} \qquad (2.80)$$

CHAPTER III

RESULTS AND DISCUSSIONS

In the previous chapter, a set of vibration problems has been reduced to that of solving second order nonlinear ordinary differential equations. In these equations non-dimensional time occurs as the independent variable. These equations have been solved by numerical integrations using the Runge-Kutta-Gill method. Computations have been carried out on IBM 7044. The Poisson's ratio has been taken to be 0.3 in all the calculations.

Table 1 contains the following results for square plates vibrating in the fundamental mode :

i) periods given by the classical theory (T_C)

ii) Mindlin's values*(Ref.63) for linear periods with the effects of transverse shear and rotatory inertia

iii) linear periods (T_L) with both the above effects included

iv) nonlinear periods (T_N).

---

* Time periods have been computed from the equations given by Mindlin and shear coefficient has been taken to be $\pi^2/12$ as derived in Ref. 61.

this approach are a little higher.

Table 4 gives the values of periods of a beam of rectangular section for different thicknesses. The following results are given :

i) periods given by the classical theory

ii) periods for the linear case obtained from Timoshenko's equation[2] taking shear coefficient equal to $\pi^2/12$ for the fundamental mode as derived in Ref. 9.

iii) periods for the linear case (T_L) as obtained in the present work

iv) periods for the nonlinear case (T_N).

There is a close agreement between the linear periods obtained here and those obtained from Timoshenko's equations. Fig. 8 shows a plot of T_N/T_C as obtained in the present work. As in the case of plates the nonlinearity in this case also is of hardening type. For comparison the curve given by Chu and Herrmann[17] for thin beams, is also plotted*. The agreement between their results and the present results is good for thin beams (H_r = 30 or more).

Tables 5 to 9 list the values of periods for the Doubly curved shell discussed in section 2.3 of Chapter II. The values are for different initial rise of the shell on square plan and for various thicknesses.

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* Perturbation solution given by Chu and Herrmann has been taken.

Values for various amplitude ratios are listed. Corresponding linear periods are also given in all the tables. Comparing Table 5 with Tables 6 through 9 it is seen that nonlinearity is of hardening type (similar to that of a flat plate) for a shell of slight initial rise ($k = 0.2$) whereas the nonlinearity is of softening type for shells of sufficient initial rise ($k = 1, 2, 3$ etc.). The elements for which nonlinearity is of hardening type are termed slightly curved elements and those for which nonlinearity is of softening type are termed shells. In the present case the transition from a slightly curved element to a shell takes place for k somewhere between 0.2 and 1.0. The phenomenon of change of nonlinearity from hardening to softening can be explained from the governing equation for plate vibration, eq. (2.16) and that for the shell vibration, eq. (2.52). The P^2 term which is primarily responsible for changing nonlinearity from hardening to softening is present in the latter equation. This term is essentially associated with curvature terms and vanishes for flat elements. This has not been pointed out by Chauhan and Ashwell[19], who studied vibration of thin Doubly Curved shells and flat plates.

Figures 9 through 12 show the plot of the amplitude ratio versus period for shells of different initial rise. It is noticed that unlike in the case of a flat element the positive (outward) and negative (inward) amplitudes of vibration for curved elements are not the same. The negative amplitude is always larger. Dynamic snap-through is clearly seen from the figures. Before snap-through occurs the shells have nonlinearity of the softening type whereas after the snap-through the nonlinearity changes to hardening type. Results for thin shallow shells of similar geometry are given in Ref. 19 in the form of graphs. The trend of results is similar to that obtained in the present work. An accurate comparison with the present results is not possible because no tables of values are given. However, some limiting values have been compared and the comparison is good. For example, for a thin shell (say H_r = 30) on square plan and with k = 1, the linear time period, T_L in the present work is 15.805. The corresponding $\bar{p}$ (the frequency parameter plotted on x-axis in Ref.19) is 39.411*, which is nearly the value read on Fig. 6 of the said reference. For thicker shells, say H_r = 20 and 10 the corresponding period

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* Period T in the present work is related to $\bar{p}$ as

$$T = \frac{4\pi\sqrt{3(1-\nu^2)}}{\bar{p} \cdot H}$$

T_L as converted from this $\bar{p}$ will be 10.536 , 5.268 which deviate from the present values of 10.560 , 5.352 respectively. The deviation is more for the thicker element. Similar comparison can be made for a shell on square plan with k = 2 and k = 3 for linear periods and also for periods at some other amplitudes.

Tables 10 and 11 list the values of periods for different initial rise and various thicknesses of the Doubly Curved shell on a rectangular plan (B = 0.5). The trend is similar to that for a shell on square plan.

Tables 12 through 15 list the values of periods for different initial rise and various thicknesses of shallow arch. Figures 13 through 16 show their plots. As expected the behaviour of an arch is similar to that of the Doubly Curved shell discussed earlier. It is seen that the transition from a slightly curved beam to an arch takes place below initial rise equal to the thickness of the arch (k = 1). Dynamic snap-through is clearly seen from the figures.

Values of the integrals X_1 through X_{21}, defined in section 2.5 dealing with spherical cap problem, are given in Table 16. Results for spherical caps are given in Tables 17 through 19 and those for a circular plate in Table 20. Figures 17 and 18 show plots of period

versus amplitude ratio for spherical caps of $R^* = 0.1$ and 0.2 respectively. Shallow spherical caps behave in a way similar to other shallow elements discussed earlier. The fact that the positive negative amplitudes of vibration are different in spherical caps as in other shallow elements has not been pointed out by Grossman et. al.[47] who have given results for thin spherical caps. Also, they have not discussed the phenomenon of dynamic snap-through. Figure 19 shows a plot of T_L/T_{LP} versus R^* for spherical caps (T_{LP} stands for linear period of a circular plate). In this figure a curve due to Grossman et.al. is also given.[+] For $H_r = 30$, the curve based on present results almost completely coincides with their curve. Figure 20 shows a plot of T_N/T_C versus A_R for a circular plate. Nonlinearity is of hardening type and behaviour of a circular plate is similar to that of a rectangular plate. The transition from a slightly curved plate to a shallow shell occurs at some value of R^* below 0.1.

[+] The graph has been plotted here after suitably transforming the x and y variables of the curve given by Grossman et.al.

CHAPTER IV

CONCLUSIONS

i) It is known that in the case of small amplitude flexural vibrations, transverse shear and rotatory inertia decrease the frequency. This trend continues to be so even for large amplitude vibrations. This is seen for all the structural elements analysed in this work namely, straight and curved beams, rectangular and circular plates, a Doubly Curved shallow shell and a spherical cap.

ii) For any flat element like beams and plates, the nonlinearity is of hardening type (that is, the period decreases as amplitude increases) whereas for shallow elements like shells and arches, the nonlinearity is of softening type (the period increases with amplitude). A distinction can be made between a slightly curved element and a shallow element by determining the value of the initial rise beyond which the type of nonlinearity changes from hardening to softening. This happens because of the presence of quadratic nonlinearity term in the final equation for curved elements.

iii) In a shallow element, when some initial amplitude is given, it vibrates about the initial equilibrium position and the positive amplitude (in the direction of the initial rise) is always smaller than the negative amplitude. The nonlinearity is of softening type.

When a critical value of initial amplitude is given to the shallow element, it snaps through to the other equilibrium position and stays snapped inside out. At this stage, the frequency of transverse vibration becomes zero (that is, the time period becomes infinite).

When an amplitude of magnitude greater than the critical value is given, it passes through both the equilibrium positions during its oscillations and the negative amplitude is very large. The nonlinearity changes from softening to hardening type. It is assumed that the material remains elastic throughout.

iv) The period of vibration decreases as initial rise of a curved element increases.

v) Of particular interest is the feature that the nonlinear effect counteracts the curvature effect. It has already been observed that the period of vibration of a curved element is less as compared to that of a flat element.

The percentage decrease is less if the nonlinear theory rather than the linear theory is employed.*

For example, a straight beam and a shallow arch of initial rise equal to four times the thickness (that is, $k = 4$) are considered. Ratio of length of span to thickness in both the cases is taken to be 20 (that is, $H_r = 20$) and the nonlinear periods of vibration for an amplitude ratio of 1.0 are considered. The period based on nonlinear theory decreases from 24.77 to 4.80 whereas that based on the linear theory decreases from 44.29 to 4.49. The percentage decrease in the period given by the nonlinear theory is much less than that in the period given by the linear theory.

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* The comparison of periods is made before the dynamic snap-through occurs.

CHAPTER V

RECOMMENDATIONS FOR FURTHER WORK

i) In the present work the transverse deformation has been taken as :

$$w(x,y,z,t) = w(x,y,t).$$

Thus the effect of normal strain is neglected which is justified because the fundamental frequency of flexural vibration is much less than the frequency of thickness-stretch mode of vibration. However, the effect of normal strain can be included by taking w in the form :

$$w(x,y,z,t) = w_0(x,y,t) + z.\psi(x,y,t)$$

ii) Sine functions have been assumed in this work for the initial unstressed shape of the Doubly Curved shallow shell and also for the shallow arch. Different geometries can however, be analysed. The effects of imperfections in the initial geometry on phenomena like dynamic buckling etc. can also be studied.

iii) In this work, elements with hinged ends have been analysed. Extension can be made to other boundary conditions.

iv) The coupling of modes between the fundamental and higher ones has been neglected in the present work. The mode shape can be refined by taking additional terms. For example, for a hinged beam vibrating in the fundamental mode, the transverse deflection can be considered as :

$$w = P(t) \left[\sin \frac{\pi x}{a} + p_1 \sin \frac{3\pi x}{a} + ---- \right]$$

and similarly for other components of displacement and rotation. The additional unknowns p_1, ------ etc. can be solved from the additional equations obtained from variation; that is, by equating the coefficients of δp_1, ------ etc. to zero.

v) Free vibration problems have been analysed here, however, forced vibration problems can also be studied in a similar way.

vi) Extension of the present work to deep shells is feasible. In that case, the displacement and rotation functions should be selected judiciously and any integration to be carried over the surface of the shell should not be carried over the projected plan.

vii) If the amplitude of vibration is not small, some materials may exhibit nonlinear elastic properties. In such a case, the present work should be extended to include both geometric and material nonlinearities.

viii) In this work, the structural material has been assumed to be isotropic. However, anisotropic materials are used widely in the modern age. Extension of this work for such materials is possible by taking suitable constitutive equations.

TABLE 1

PERIODS FOR A SQUARE PLATE (B=1)

E_r	T_C	T^*_M	T_L	T_N									
				AMPLITUDE RATIO(A_R)									
				0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	42.075	42.152	42.170	41.93	41.25	38.83	35.62	32.25	29.07	22.66	18.22	15.11	12.85
30	31.555	31.684	31.684	31.51	30.99	29.17	26.75	24.21	21.82	17.01	13.67	11.33	9.64
20	21.037	21.235	21.229	21.11	20.76	19.53	17.91	16.20	14.59	11.37	9.13	7.56	6.44
10	10.519	10.897	10.900	10.84	10.65	10.00	9.14	8.25	7.42	5.76	4.62	3.83	3.26

*T_M - linear periods, including the effects of transverse shear and rotatory inertia, computed from expressions given by Mindlin (see Chapter III).

TABLE 2

PERIODS FOR A RECTANGULAR PLATE (B=0.5)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	16.926	16.80	16.43	15.17	13.60	12.07	10.70	8.10	6.42	5.28	4.47
30	12.750	12.65	12.37	11.42	10.24	9.08	8.04	6.10	4.82	3.97	3.36
20	8.606	8.54	8.34	7.69	6.89	6.10	5.40	4.08	3.23	2.66	2.25
10	4.585	4.54	4.44	4.06	3.62	3.18	2.80	2.12	1.68	1.39	1.19

TABLE 3

PERIODS FOR A SQUARE PLATE
(BY ERRONEOUS USE OF GALERKIN TECHNIQUE)

H_r	T_C	T_L	T_N					
			AMPLITUDE RATIO (A_R)					
			0.1	0.2	0.4	0.6	0.8	1.0
40	42.075	42.170	41.96	41.36	39.20	36.26	33.10	30.04
30	31.556	31.684	31.53	31.08	29.45	27.23	24.85	22.56
20	21.037	21.229	21.12	20.82	19.72	18.23	16.63	15.09
10	10.519	10.900	10.84	10.68	10.10	9.31	8.47	7.67

TABLE 4

PERIODS FOR A BEAM

H_r	T_C	T_T^*	T_L	T_N									
				AMPLITUDE RATIO (A_R)									
				0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	88.212	88.032	88.306	87.33	84.58	75.82	65.89	56.93	49.46	36.39	28.40	23.17	19.53
30	66.16[illegible]	66.259	66.284	65.54	63.49	56.90	49.45	42.72	37.11	27.30	21.31	17.38	14.64
20	44.106	44.289	44.293	43.80	42.42	38.01	33.02	28.50	24.77	18.22	14.22	11.60	9.77
10	22.053	22.425	22.424	22.17	21.46	19.19	16.65	14.36	12.46	9.15	7.14	5.82	4.90

*T_T - Linear periods including the effects of transverse shear and rotatory inertia, computed from expressions given by Timoshenko (taking shear coefficient to be $\pi^2/12$ — see Chapter III).

TABLE 5

PERIODS FOR A D.C. (DOUBLY CURVED) SHELL(B=1;k=.0.2)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	39.840	39.74 (0.106)*	39.39 (0.223)	37.59 (0.486)	34.37 (0.758)	30.75 (1.018)	27.45 (1.261)	21.20 (1.821)	17.08 (2.352)	14.25 (2.868)	12.19 (3.378)
30	29.928	29.85 (0.106)	29.59 (0.223)	28.23 (0.486)	25.81 (0.759)	23.09 (1.018)	20.61 (1.261)	15.91 (1.822)	12.82 (2.351)	10.69 (2.866)	9.15 (3.375)
20	20.044	19.99 (0.106)	19.81 (0.223)	18.90 (0.486)	17.27 (0.760)	15.44 (1.019)	13.77 (1.262)	10.63 (1.822)	8.57 (2.351)	7.14 (2.866)	6.11 (3.376)
10	10.269	10.24 (0.106)	10.15 (0.224)	9.67 (0.490)	8.81 (0.765)	7.85 (1.024)	6.99 (1.267)	5.38 (1.826)	4.33 (2.354)	3.61 (2.869)	3.10 (3.378)

*In this and subsequent tables, figures in brackets given below the values of T_N indicate corresponding negative (inward) amplitude ratio.

TABLE 6

PERIODS FOR A D.C. SHELL (B=1;k=1)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	21.058	21.15 (0.108)	21.45 (0.236)	23.24 (0.583)	29.67 (1.232)	29.52 (2.095)	24.47 (2.532)	17.80 (3.250)	14.31 (3.838)	12.03 (4.385)	10.41 (4.913)
30	15.805	15.87 (0.108)	16.10 (0.236)	17.43 (0.583)	22.30 (1.234)	22.14 (2.097)	18.36 (2.533)	13.37 (3.251)	10.73 (3.838)	9.03 (4.385)	7.82 (4.914)
20	10.560	10.61 (0.108)	10.76 (0.236)	11.65 (0.584)	14.97 (1.240)	14.81 (2.104)	12.26 (2.537)	8.93 (3.253)	7.17 (3.840)	6.03 (4.386)	5.22 (4.915)
10	5.352	5.38 (0.108)	5.46 (0.237)	5.94 (0.591)	7.81 (1.286)	7.47 (2.147)	6.16 (2.566)	4.50 (3.270)	3.61 (3.853)	3.05 (4.397)	2.64 (4.924)

TABLE 7

PERIODS FOR A D.C. SHELL (B=1; k=2)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	11.678	11.69 (0.105)	11.76 (0.220)	12.03 (0.493)	12.77 (0.848)	14.70 (1.380)	30.28 (4.304)	16.42 (5.158)	12.72 (5.775)	10.62 (6.336)	9.20 (6.874)
30	8.763	8.77 (0.105)	8.83 (0.220)	9.05 (0.493)	9.59 (0.848)	11.03 (1.380)	22.68 (4.307)	12.32 (5.160)	9.54 (5.775)	7.97 (6.336)	6.90 (6.874)
20	5.854	5.87 (0.105)	5.89 (0.221)	6.05 (0.493)	6.41 (0.849)	7.38 (1.384)	15.05 (4.317)	8.22 (5.164)	6.37 (5.778)	5.31 (6.339)	4.61 (6.876)
10	2.978	2.98 (0.105)	3.00 (0.222)	3.08 (0.498)	3.27 (0.865)	3.82 (1.437)	7.12 (4.392)	4.09 (5.197)	3.19 (5.802)	2.67 (6.359)	2.32 (6.893)

TABLE 8

PERIODS FOR A D.C. SHELL (B=1; k=3)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	7.957	7.96 (0.103)	7.98 (0.214)	8.07 (0.460)	8.25 (0.748)	8.57 (1.095)	9.18 (1.540)	17.81 (7.111)	12.23 (7.741)	9.93 (8.309)	8.52 (8.851)
30	5.971	5.97 (0.103)	5.99 (0.214)	6.06 (0.460)	6.19 (0.748)	6.43 (1.096)	6.89 (1.541)	13.35 (7.112)	9.17 (7.742)	7.45 (8.310)	6.39 (8.852)
20	3.991	4.00 (0.103)	4.01 (0.214)	4.05 (0.460)	4.14 (0.749)	4.30 (1.098)	4.61 (1.546)	8.87 (7.118)	6.11 (7.746)	4.97 (8.313)	4.26 (8.855)
10	2.053	2.05 (0.104)	2.06 (0.215)	2.08 (0.466)	2.13 (0.765)	2.22 (1.135)	2.40 (1.627)	4.22 (7.170)	3.01 (7.783)	2.47 (8.343)	2.13 (8.880)

TABLE 9

PERIODS FOR A D.C. SHELL (B=1; k=4)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	6.015	6.02 (0.103)	6.03 (0.210)	6.06 (0.444)	6.13 (0.705)	6.25 (1.001)	6.42 (1.343)	7.66 (2.601)	12.99 (9.721)	9.76 (10.292)	8.18 (10.837)
30	4.515	4.52 (0.103)	4.52 (0.210)	4.55 (0.444)	4.60 (0.705)	4.69 (1.002)	4.82 (1.344)	5.75 (2.604)	9.73 (9.722)	7.31 (10.293)	6.13 (10.838)
20	3.020	3.02 (0.103)	3.03 (0.210)	3.04 (0.444)	3.08 (0.706)	3.14 (1.004)	3.23 (1.348)	3.86 (2.624)	6.46 (9.727)	4.87 (10.297)	4.08 (10.841)
10	1.578	1.58 (0.103)	1.58 (0.212)	1.59 (0.452)	1.61 (0.725)	1.64 (1.043)	1.69 (1.421)	2.18 (3.020)	3.02 (9.777)	2.37 (10.337)	2.02 (10.875)

TABLE 10

PERIODS FOR A D.C. SHELL (B=0.5; k=1)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	7.484	7.52 (0.109)	7.65 (0.239)	8.48 (0.609)	13.31 (1.525)	10.46 (2.303)	8.54 (2.661)	6.19 (3.317)	4.96 (3.881)	4.16 (4.415)	3.60 (4.936)
30	5.623	5.65 (0.109)	5.75 (0.239)	6.38 (0.610)	10.06 (1.536)	7.85 (2.308)	6.42 (2.664)	4.64 (3.318)	3.72 (3.882)	3.12 (4.416)	2.72 (4.936)
20	3.771	3.79 (0.109)	3.86 (0.239)	4.28 (0.613)	6.86 (1.574)	5.24 (2.322)	4.29 (2.674)	3.11 (3.324)	2.49 (3.886)	2.10 (4.420)	1.81 (4.940)
10	1.968	1.98 (0.110)	2.02 (0.245)	2.28 (0.653)	3.60 (1.863)	2.61 (2.424)	2.15 (2.754)	1.58 (3.382)	1.30 (3.938)	1.10 (4.471)	0.98 (4.993)

TABLE 11

PERIODS FOR A D.C.SHELL (B=0.5;k=4)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	2.073	2.07 (0.103)	2.08 (0.210)	2.09 (0.444)	2.11 (0.706)	2.15 (1.004)	2.21 (1.348)	2.67 (2.631)	4.41 (9.800)	3.34 (10.350)	2.80 (10.882)
30	1.561	1.56 (0.103)	1.56 (0.211)	1.57 (0.445)	1.59 (0.708)	1.62 (1.007)	1.67 (1.355)	2.02 (2.660)	3.29 (9.804)	2.50 (10.354)	2.10 (10.885)
20	1.063	1.06 (0.103)	1.07 (0.211)	1.07 (0.448)	1.08 (0.717)	1.10 (1.025)	1.14 (1.389)	1.42 (2.836)	2.12 (9.822)	1.64 (10.369)	1.39 (10.896)
10	0.681	0.68 (0.105)	0.68 (0.220)	0.69 (0.488)	0.69 (0.826)	0.71 (1.275)	0.75 (1.946)	1.22 (9.375)	0.95 (9.915)	0.83 (10.437)	0.75 (10.949)

TABLE 12

PERIODS FOR A SHALLOW ARCH (k=1)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	33.354	33.55 (0.109)	34.25 (0.242)	39.21 (0.639)	65.83 (2.004)	45.65 (2.473)	37.34 (2.770)	23.98 (3.374)	21.56 (3.918)	18.07 (4.441)	15.61 (4.956)
30	25.023	25.17 (0.109)	25.69 (0.242)	29.42 (0.639)	49.37 (2.005)	34.25 (2.474)	28.01 (2.771)	20.24 (3.374)	16.17 (3.918)	13.56 (4.442)	11.71 (4.956)
20	16.696	16.79 (0.109)	17.15 (0.242)	19.64 (0.639)	32.92 (2.008)	22.84 (2.475)	18.69 (2.772)	13.50 (3.374)	10.79 (3.918)	9.05 (4.442)	7.81 (4.956)
10	8.389	8.44 (0.109)	8.62 (0.242)	9.89 (0.642)	16.46 (2.028)	11.45 (2.483)	9.37 (2.777)	6.77 (3.378)	5.41 (3.920)	4.54 (4.444)	3.92 (4.958)

TABLE 13

PERIODS FOR A SHALLOW ARCH (k=2)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	17.648	17.68 (0.105)	17.78 (0.221)	18.29 (0.498)	19.54 (0.865)	23.62 (1.460)	39.96 (4.703)	24.28 (5.334)	18.93 (5.888)	15.82 (6.418)	13.70 (6.937)
30	13.239	13.27 (0.105)	13.34 (0.221)	13.72 (0.498)	14.65 (0.865)	17.73 (1.460)	29.27 (4.704)	18.21 (5.334)	14.20 (5.888)	11.87 (6.418)	10.27 (6.937)
20	8.833	8.85 (0.105)	8.90 (0.221)	9.15 (0.498)	9.78 (0.866)	11.83 (1.461)	19.98 (4.705)	12.14 (5.335)	9.47 (5.889)	7.91 (6.418)	6.85 (6.937)
10	4.441	4.45 (0.105)	4.48 (0.222)	4.60 (0.499)	4.92 (0.870)	5.99 (1.477)	9.93 (4.715)	6.08 (5.340)	4.75 (5.893)	3.97 (6.422)	3.44 (6.940)

TABLE 14

PERIODS FOR A SHALLOW ARCH (k=3)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	11.898	11.91 (0.103)	11.94 (0.214)	12.08 (0.461)	12.36 (0.751)	12.87 (1.104)	13.88 (1.562)	25.58 (7.314)	18.05 (7.873)	14.72 (8.405)	12.64 (8.926)
30	8.926	8.93 (0.103)	8.96 (0.214)	9.06 (0.461)	9.27 (0.751)	9.66 (1.104)	10.41 (1.562)	19.19 (7.315)	13.54 (7.873)	11.04 (8.405)	9.48 (8.926)
20	5.956	5.96 (0.103)	5.98 (0.214)	6.04 (0.461)	6.18 (0.751)	6.44 (1.104)	6.95 (1.563)	12.79 (7.316)	9.03 (7.874)	7.36 (8.406)	6.32 (8.926)
10	3.003	3.01 (0.103)	3.01 (0.214)	3.05 (0.463)	3.11 (0.755)	3.25 (1.113)	3.51 (1.583)	6.33 (7.324)	4.50 (7.879)	3.68 (8.411)	3.17 (8.930)

TABLE 15

PERIODS FOR A SHALLOW ARCH (k=4)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	8.959	8.96 (0.103)	8.98 (0.210)	9.04 (0.444)	9.14 (0.706)	9.32 (1.004)	9.59 (1.349)	11.58 (2.636)	18.96 (9.864)	14.41 (10.397)	12.11 (10.919)
30	6.721	6.72 (0.103)	6.74 (0.210)	6.78 (0.444)	6.86 (0.706)	6.99 (1.004)	7.20 (1.349)	8.68 (2.637)	14.22 (9.864)	10.81 (10.397)	9.08 (10.919)
20	4.486	4.49 (0.103)	4.50 (0.210)	4.52 (0.444)	4.58 (0.707)	4.66 (1.004)	4.80 (1.350)	5.80 (2.642)	9.48 (9.865)	7.20 (10.398)	6.06 (10.919)
10	2.270	2.27 (0.103)	2.27 (0.211)	2.29 (0.446)	2.32 (0.711)	2.36 (1.014)	2.43 (1.367)	2.98 (2.728)	4.66 (9.873)	3.58 (10.405)	3.02 (10.925)

TABLE 16

SOME INTEGRALS IN SPHERICAL CAP PROBLEM

Integral	Value	Integral	Value
X_1	0.16959	X_{12}	0.46258
X_2	-0.20747	X_{13}	-1.07608
X_3	-0.35158	X_{14}	-0.97503
X_4	-1.11781	X_{15}	-0.16959
X_5	-1.38822	X_{16}	0.45219
X_6	0.51098	X_{17}	-0.38593
X_7	-0.93106	X_{18}	1.11810
X_8	-0.59501	X_{19}	-3.36817
X_9	0.81439	X_{20}	3.29342
X_{10}	-0.46282	X_{21}	4.65301
X_{11}	0.46553		

(λa and f have been calculated to be 2.2174 and 0.0379 respectively. These values have been used in evaluating the above integrals).

TABLE 17

PERIODS FOR A SPHERICAL CAP (R^*=0.1)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	42.203	42.26 (0.105)	42.52 (0.221)	43.70 (0.496)	46.61 (0.860)	55.98 (1.443)	96.54 (5.038)	59.24 (5.586)	46.22 (6.107)	38.61 (6.620)	33.42 (7.128)
30	41.251	41.37 (0.106)	41.77 (0.228)	43.91 (0.534)	51.18 (1.016)	93.55 (3.551)	67.41 (3.862)	46.05 (4.474)	36.48 (5.021)	30.60 (5.547)	26.50 (6.063)
20	38.858	39.04 (0.109)	39.74 (0.239)	44.16 (0.609)	79.72 (1.686)	55.85 (2.431)	45.35 (2.771)	32.70 (3.410)	26.15 (3.968)	21.95 (4.500)	18.97 (5.019)
10	30.725	30.92 (0.111)	31.59 (0.250)	34.36 (0.654)	32.80 (1.144)	28.02 (1.512)	24.20 (1.802)	18.11 (2.410)	14.52 (2.956)	12.14 (3.481)	10.43 (3.996)

TABLE 18

PERIODS FOR A SPHERICAL CAP ($R^*=0.2$)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	21.597	21.60 (0.103)	21.64 (0.210)	21.78 (0.444)	22.04 (0.707)	22.47 (1.005)	23.15 (1.351)	28.38 (2.669)	44.88 (10.699)	34.82 (11.115)	29.46 (11.557)
30	21.468	21.49 (0.103)	21.54 (0.214)	21.80 (0.461)	22.30 (0.751)	23.24 (1.104)	25.12 (1.564)	45.60 (7.911)	32.86 (8.361)	26.94 (8.830)	23.16 (9.310)
20	21.114	21.16 (0.105)	21.27 (0.221)	21.87 (0.496)	23.13 (0.860)	28.00 (1.443)	48.32 (5.035)	29.64 (5.584)	23.12 (6.106)	19.30 (6.618)	16.73 (7.126)
10	19.488	19.59 (0.109)	19.93 (0.239)	22.15 (0.610)	39.99 (1.691)	27.98 (2.431)	22.73 (2.770)	16.39 (3.410)	13.10 (3.967)	11.00 (4.499)	9.51 (5.018)

TABLE 19

PERIODS FOR A SPHERICAL CAP (R^* =0.3)

H_r	T_L	T_N									
		AMPLITUDE RATIO (A_R)									
		0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	14.462	14.46 (0.102)	14.47 (0.207)	14.51 (0.429)	14.59 (0.667)	14.69 (0.925)	14.85 (1.205)	15.53 (2.034)	17.18 (3.186)	45.30 (15.842)	29.60 (16.202)
30	14.424	14.43 (0.102)	14.44 (0.209)	14.52 (0.439)	14.66 (0.693)	14.86 (0.976)	15.19 (1.297)	17.06 (2.378)	36.68 (11.888)	26.20 (12.278)	21.82 (12.700)
20	14.319	14.33 (0.103)	14.37 (0.214)	14.55 (0.461)	14.88 (0.751)	15.50 (1.104)	16.76 (1.564)	30.42 (7.908)	21.92 (8.358)	17.96 (8.828)	15.45 (9.308)
10	13.794	13.83 (0.106)	13.97 (0.228)	14.68 (0.534)	17.12 (1.017)	31.27 (3.543)	22.52 (3.856)	15.39 (4.470)	12.19 (5.018)	10.23 (5.544)	8.86 (6.060)

TABLE 20

PERIODS FOR A CIRCULAR PLATE

H_r	T_C	T_L	T_N									
			AMPLITUDE RATIO (A_R)									
			0.1	0.2	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0
40	168.913	171.656	170.45	167.01	155.11	140.05	124.94	111.27	84.97	67.57	55.68	47.20
30	126.685	128.795	127.87	125.31	116.38	105.06	93.74	83.47	63.74	50.68	41.76	35.40
20	84.457	85.964	85.37	83.63	77.66	70.11	62.54	55.69	42.52	33.81	27.86	23.61
10	42.228	43.252	42.95	42.07	39.05	35.23	31.41	27.95	21.33	16.95	13.97	11.84

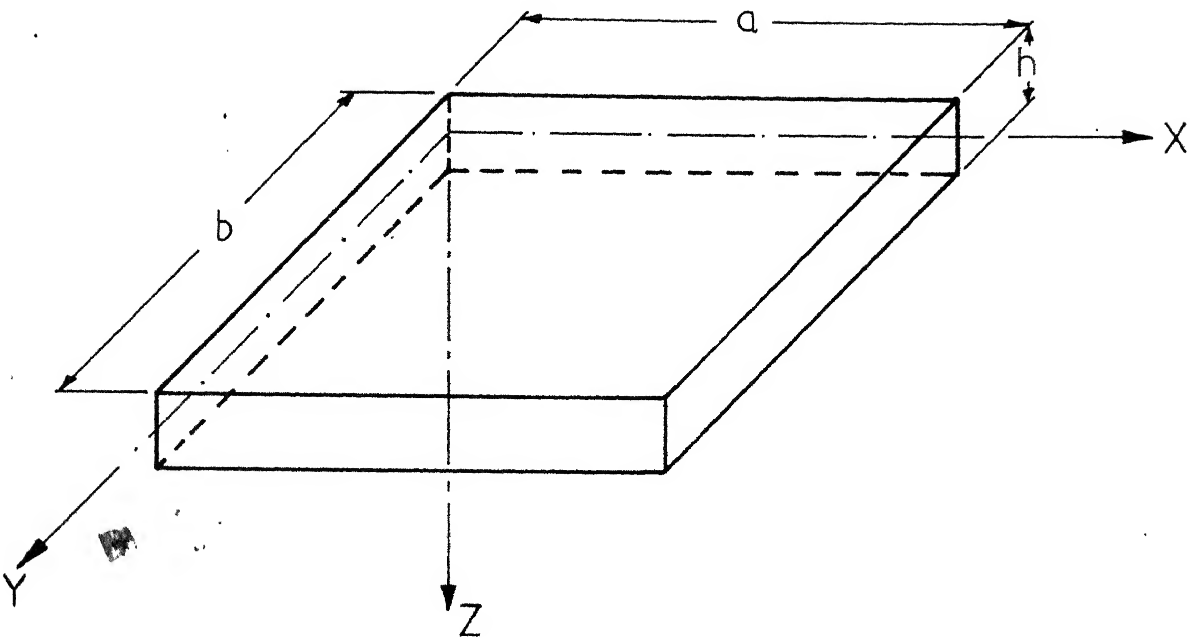

FIG. 1 _ GEOMETRY OF A RECTANGULAR PLATE

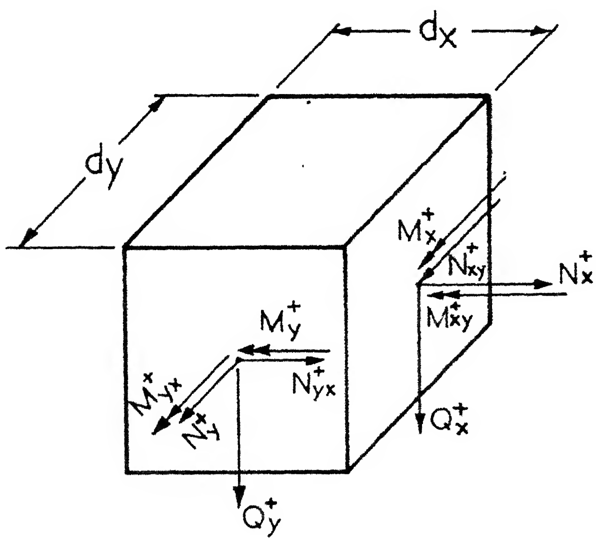

FIG. 2 _ POSITIVE SENSE OF STRESS RESULTANTS

$(N_x^+ = N_x + N_{x,x}.dx$ etc.)

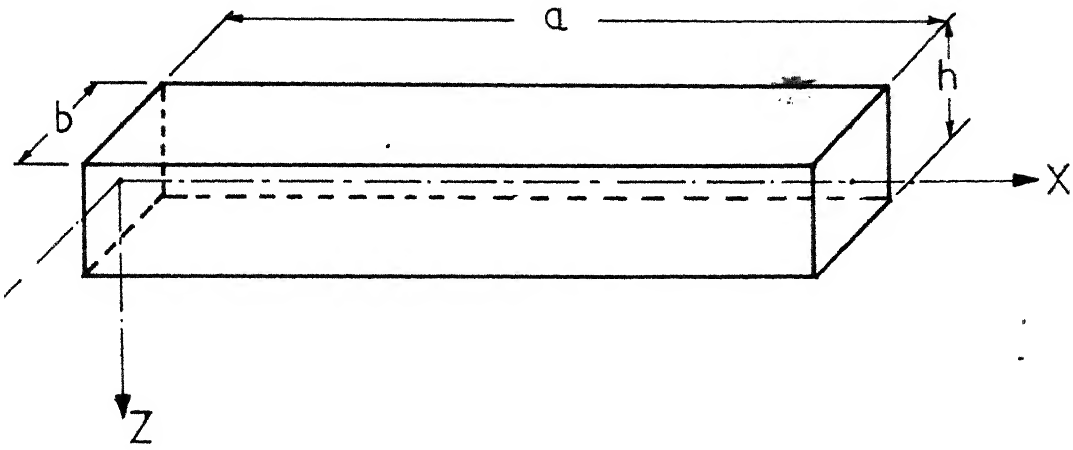

FIG.3_GEOMETRY OF A BEAM

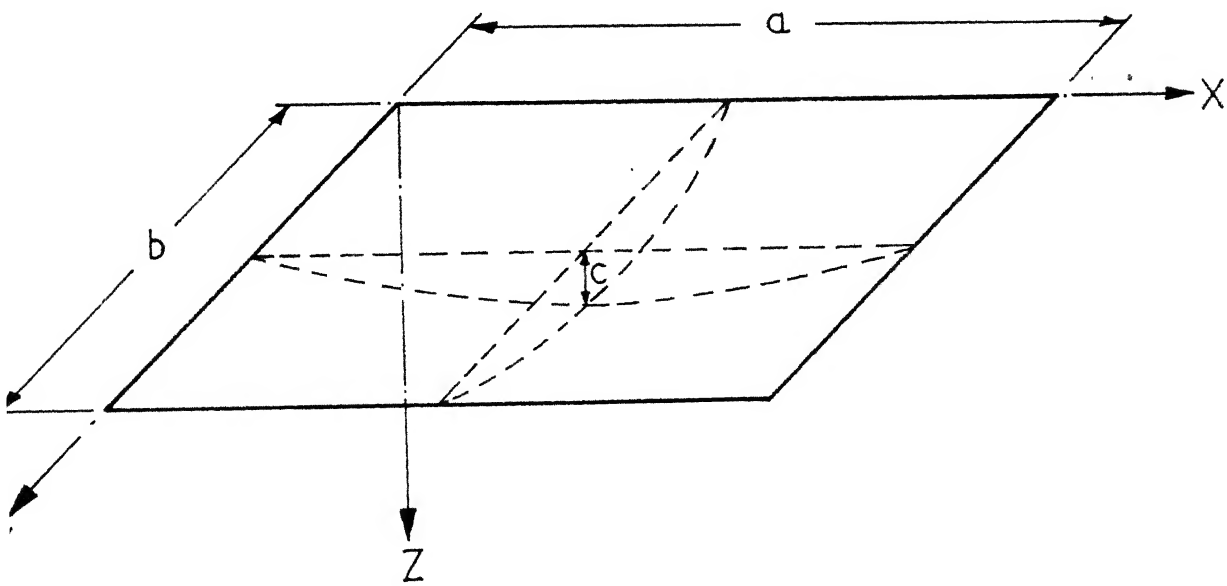

FIG. 4 _ MIDDLE SURFACE OF A DOUBLY CURVED SHELL (UNDEFORMED)

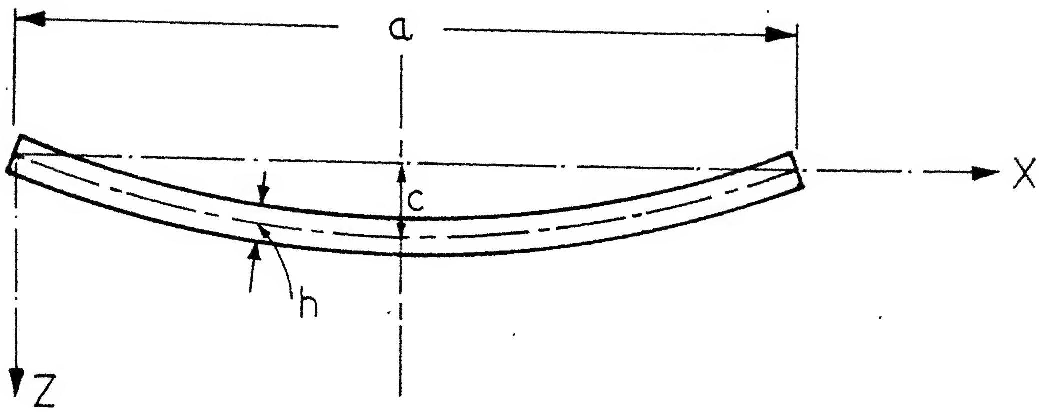

FIG. 5_ GEOMETRY OF AN ARCH (UNDEFORMED)

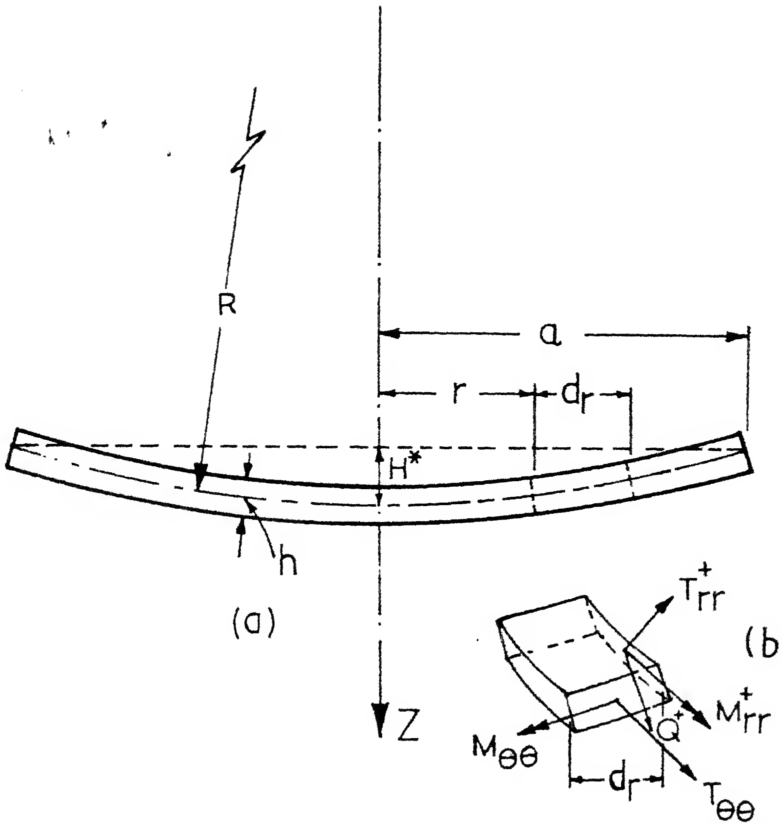

FIG.6_(a) GEOMETRY OF SPHERICAL CAP (UNDEFORMED)
(b) STRESS RESULTANTS IN SHELL ELEMENT

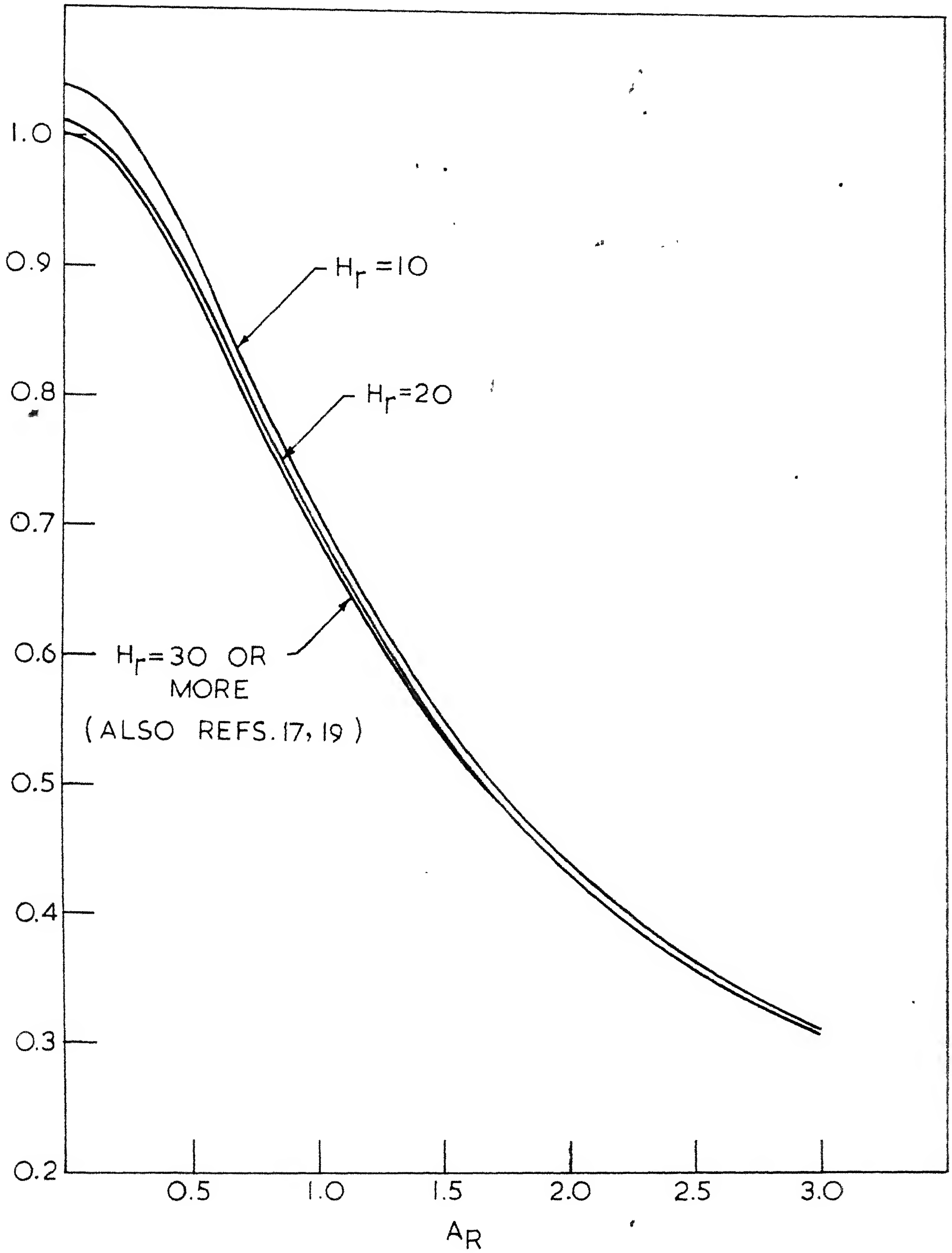

FIG. 7_ T_N / T_C VERSUS A_R FOR A SQUARE PLATE

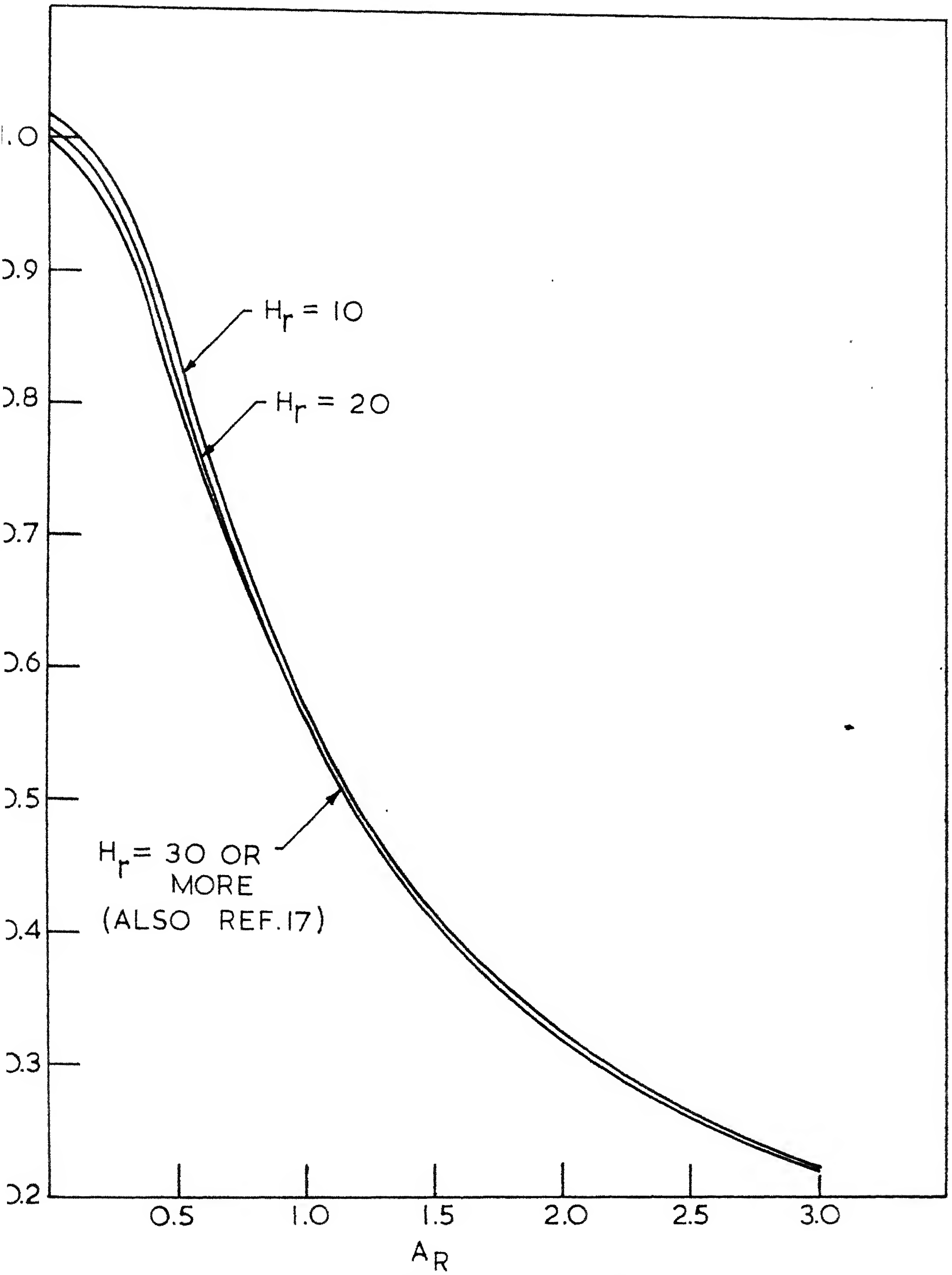

FIG.8 – T_N/T_C VERSUS A_R FOR A BEAM

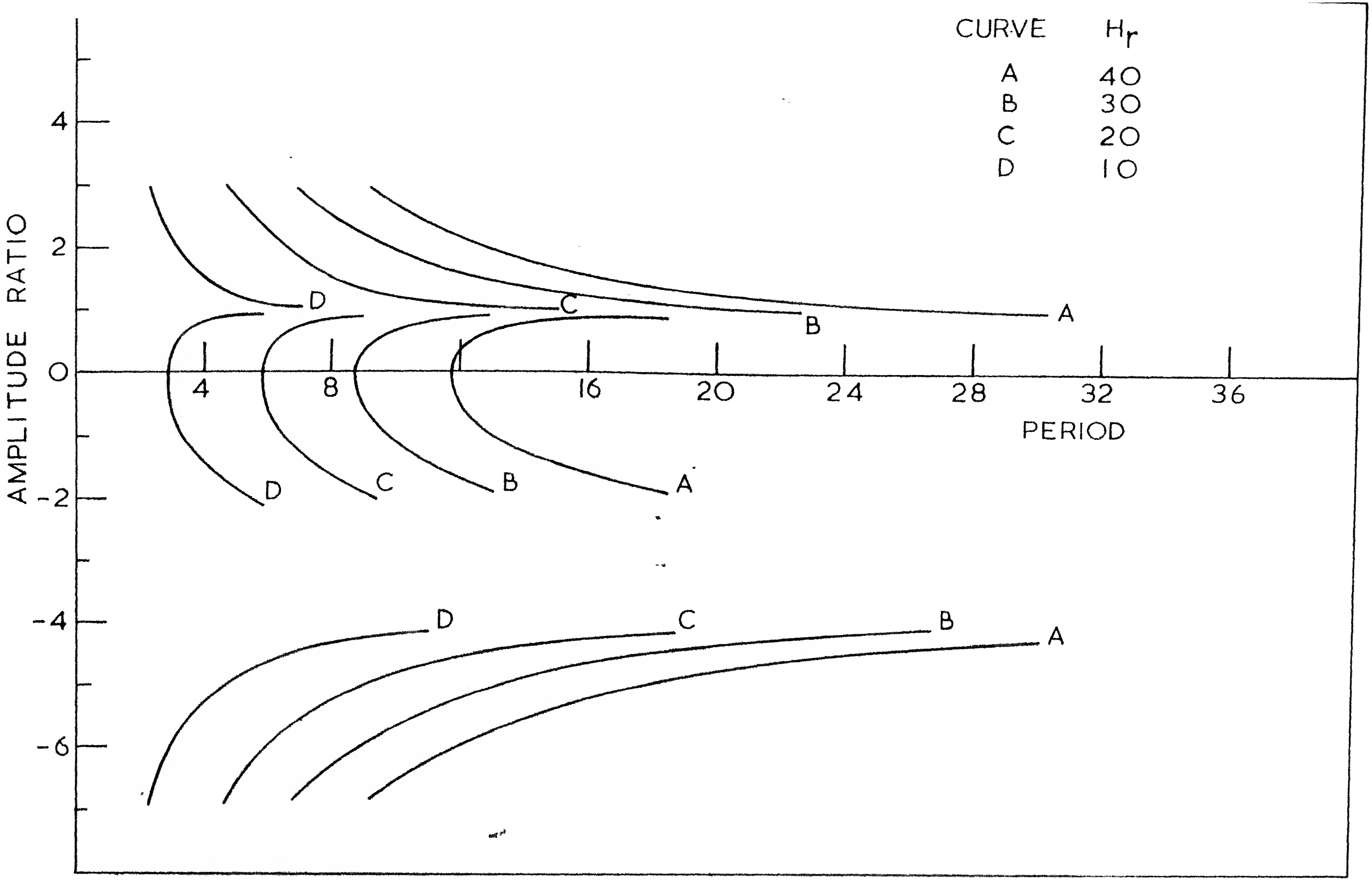

FIG.10_PERIOD VERSUS AMPLITUDE RATIO FOR A D.C. SHELL ($B = 1$; $k = 2$)

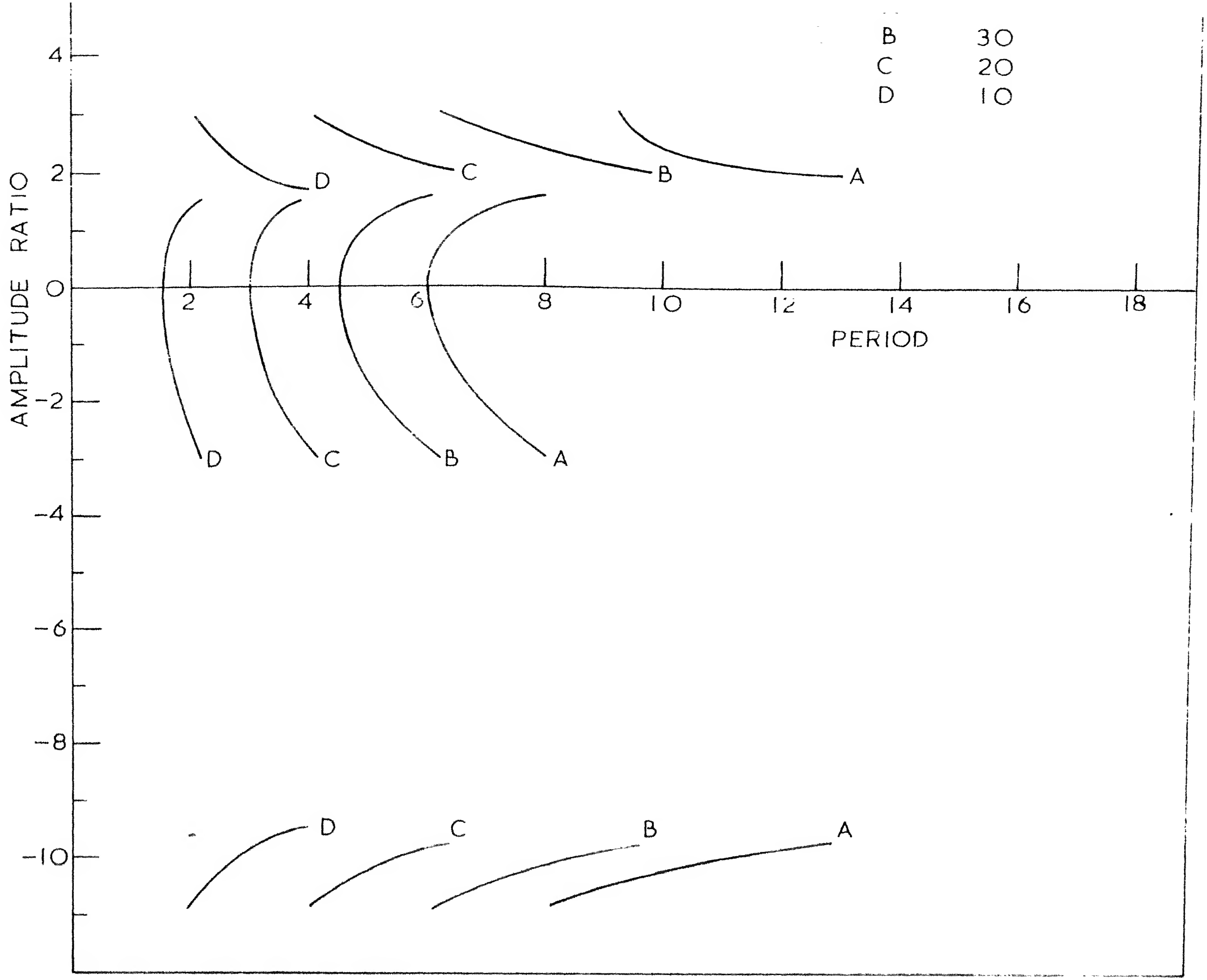

FIG.12 _ PERIOD VERSUS AMPLITUDE RATIO FOR A D.C SHELL (B = 1; k = 4)

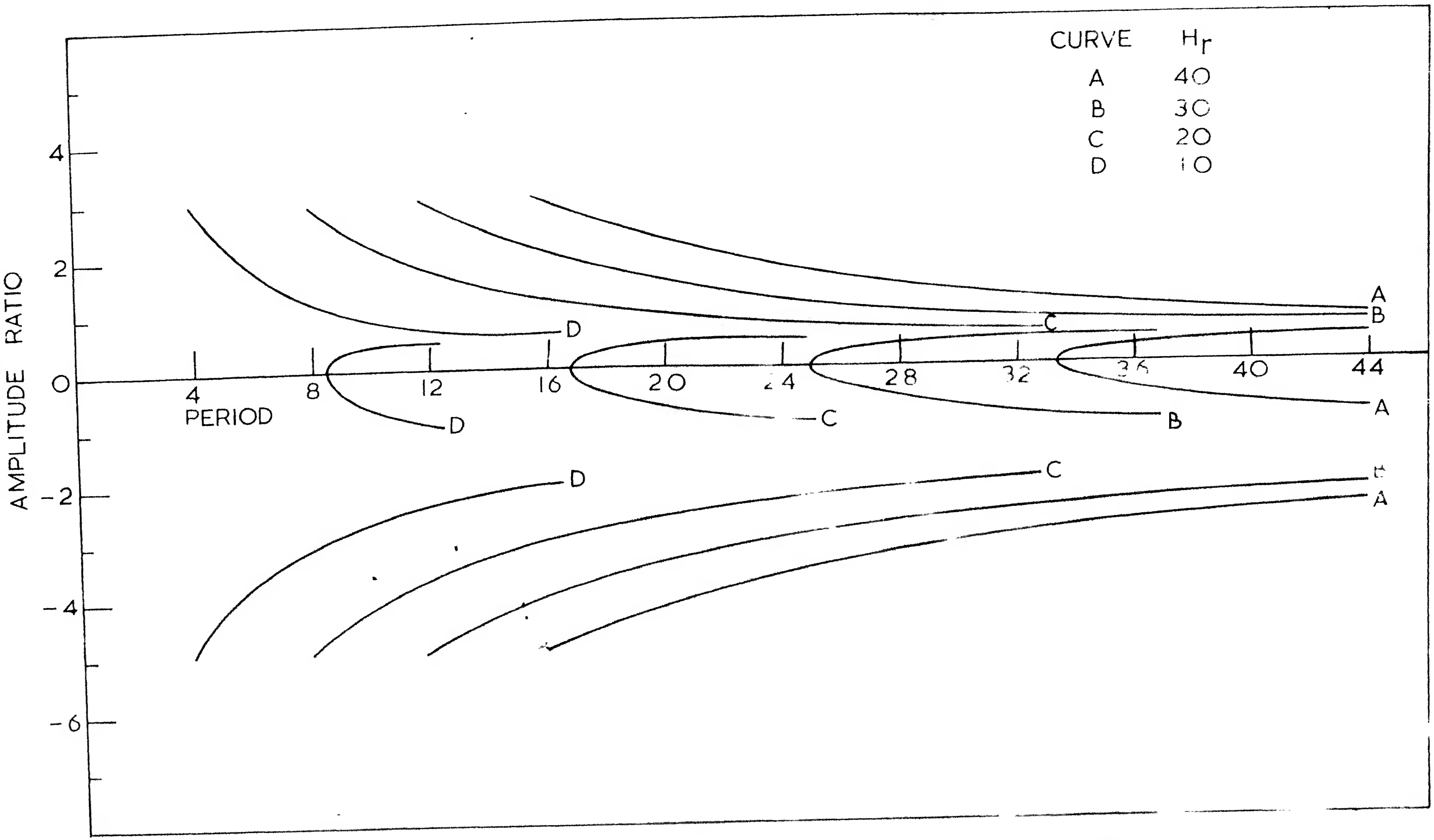

FIG.13_ PERIOD VERSUS AMPLITUDE RATIO FOR AN ARCH (k=1)

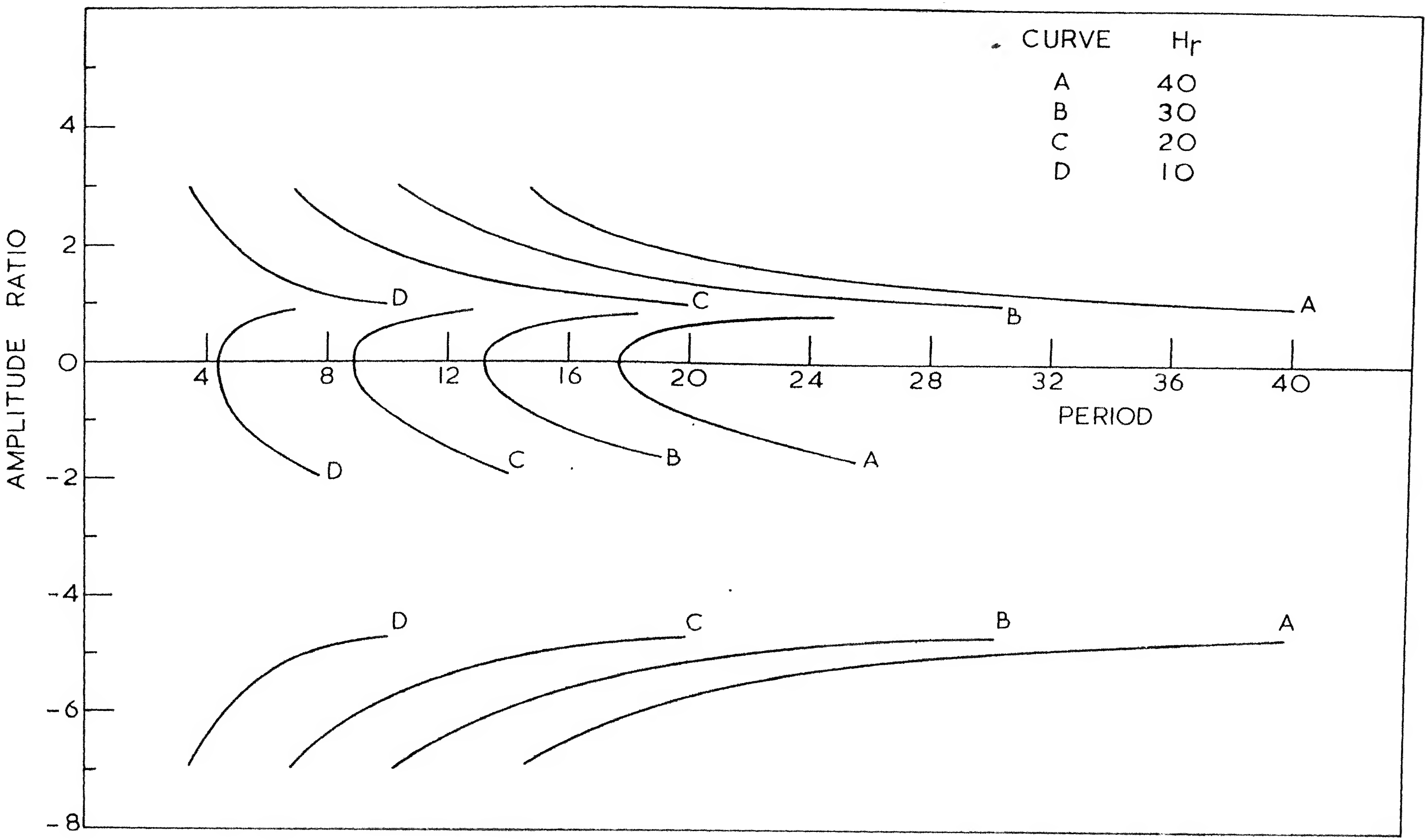

FIG.14 _ PERIOD VERSUS AMPLITUDE RATIO FOR AN ARCH (k = 2)

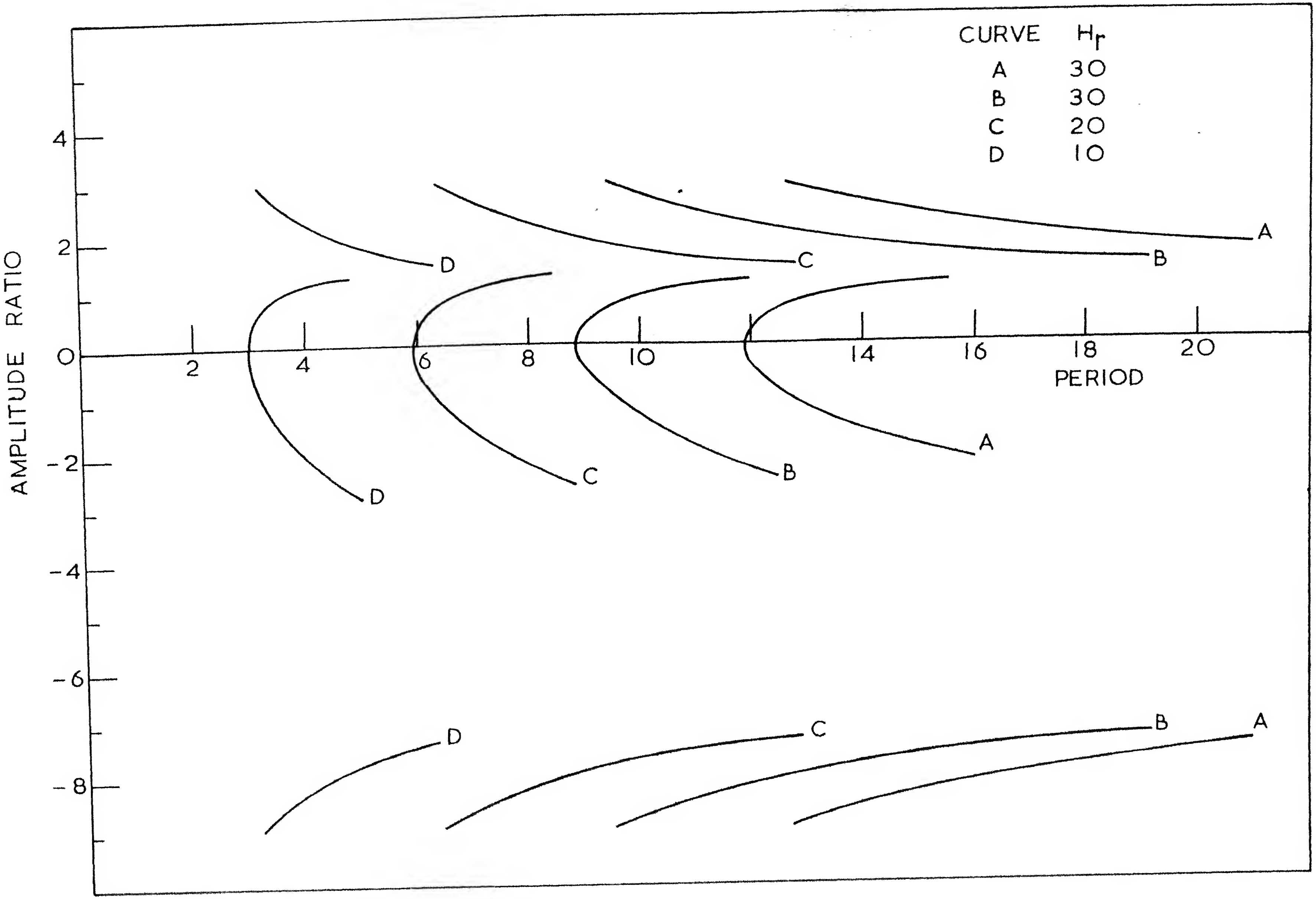

FIG.15_PERIOD VERSUS AMPLITUDE RATIO FOR AN ARCH (k=3)

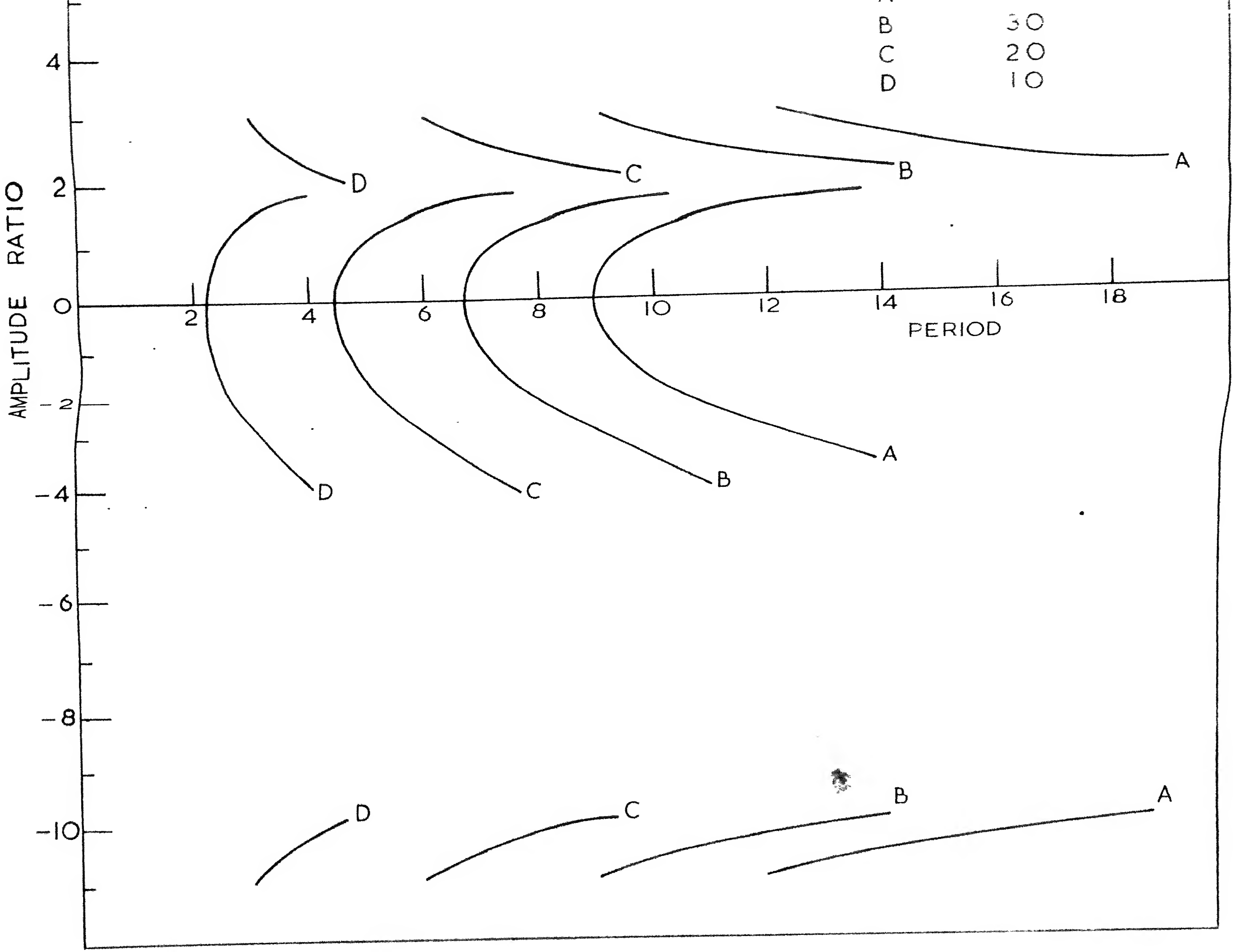

FIG.16_ PERIOD VERSUS AMPLITUDE RATIO FOR AN ARCH (k=4)

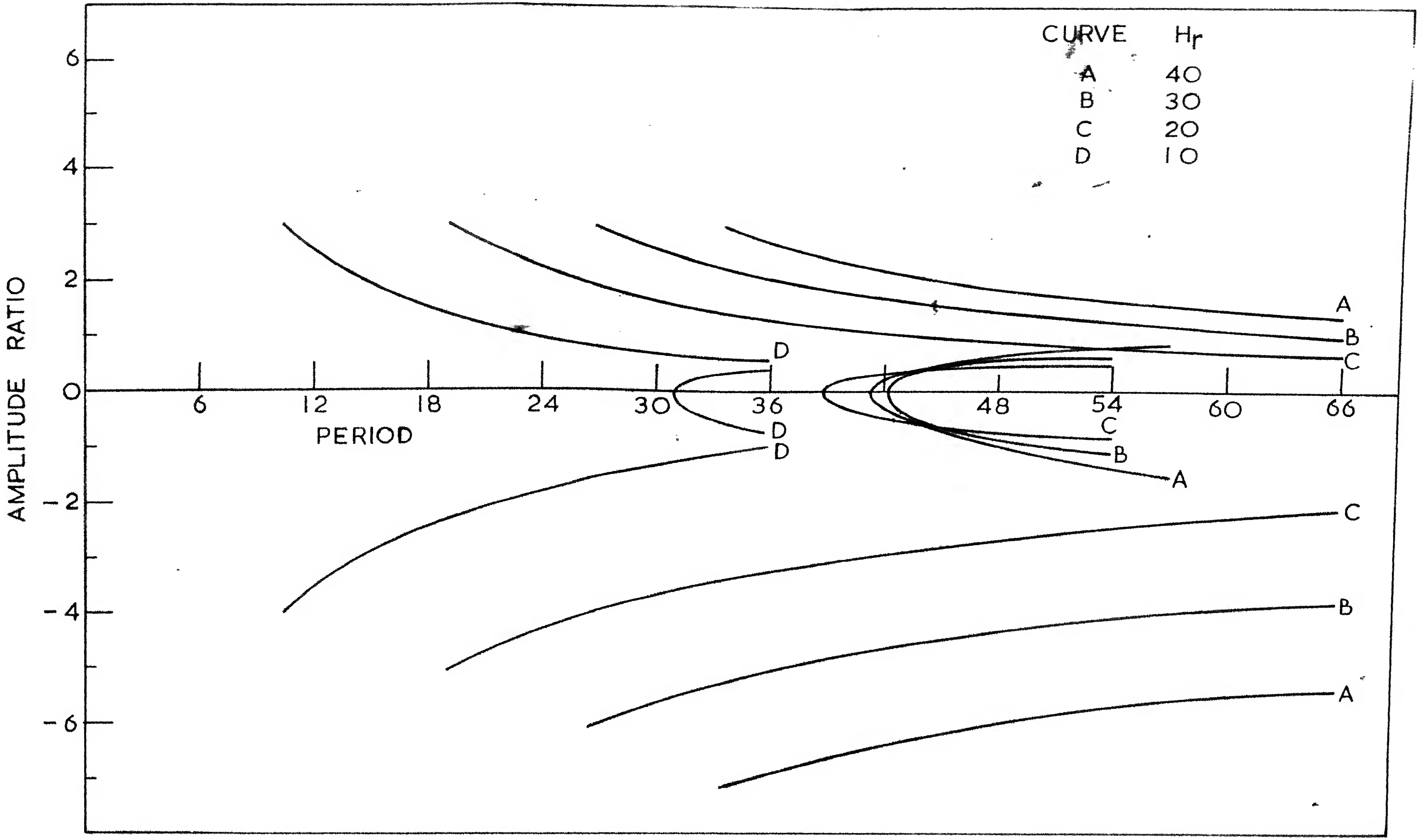

FIG.17 _ PERIOD VERSUS AMPLITUDE RATIO FOR A SPHERICAL CAP ($R^{*} = 0.1$)

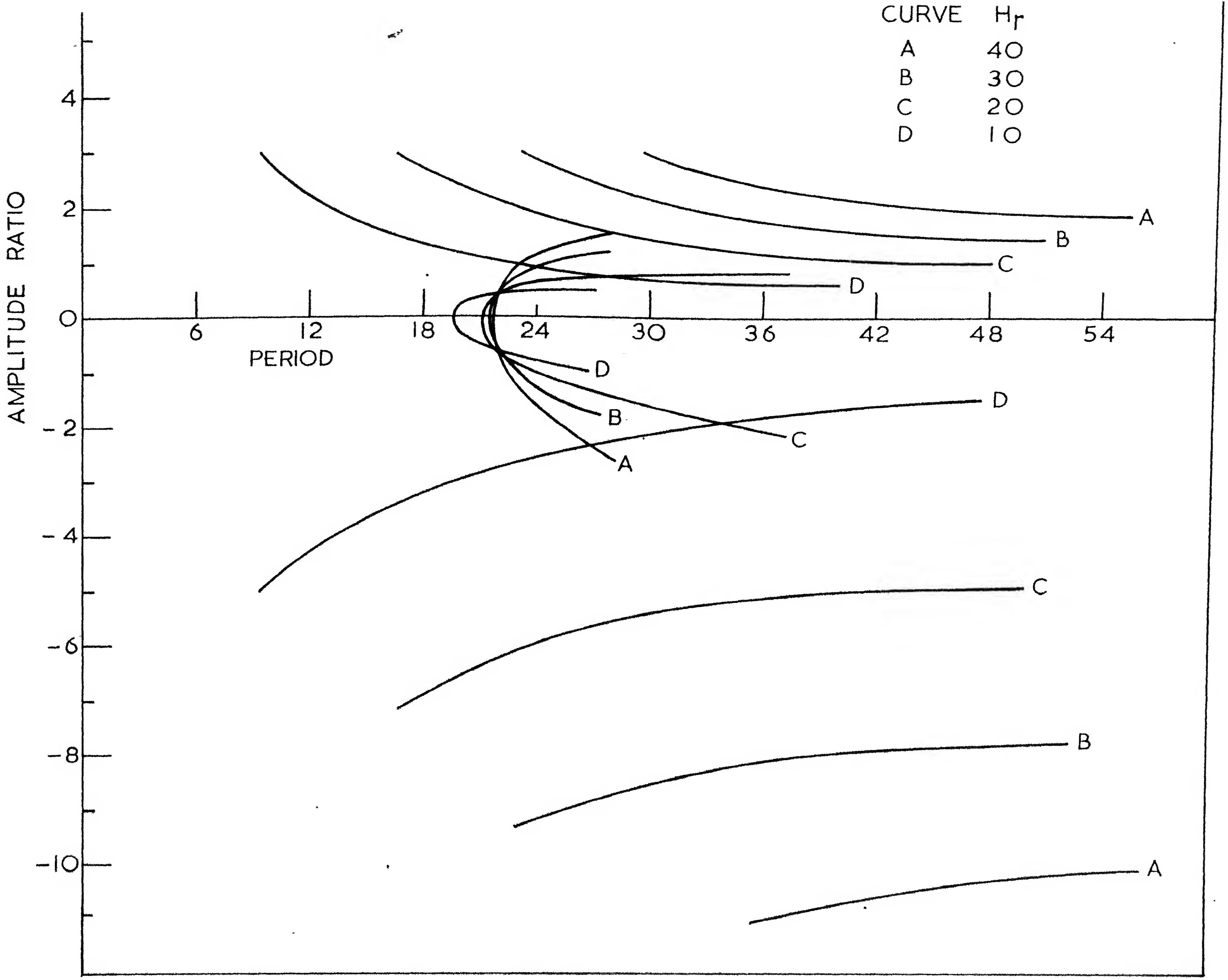

FIG.18 _ PERIOD VERSUS AMPLITUDE RATIO FOR A SPHERICAL CAP ($R^* = 0.2$)

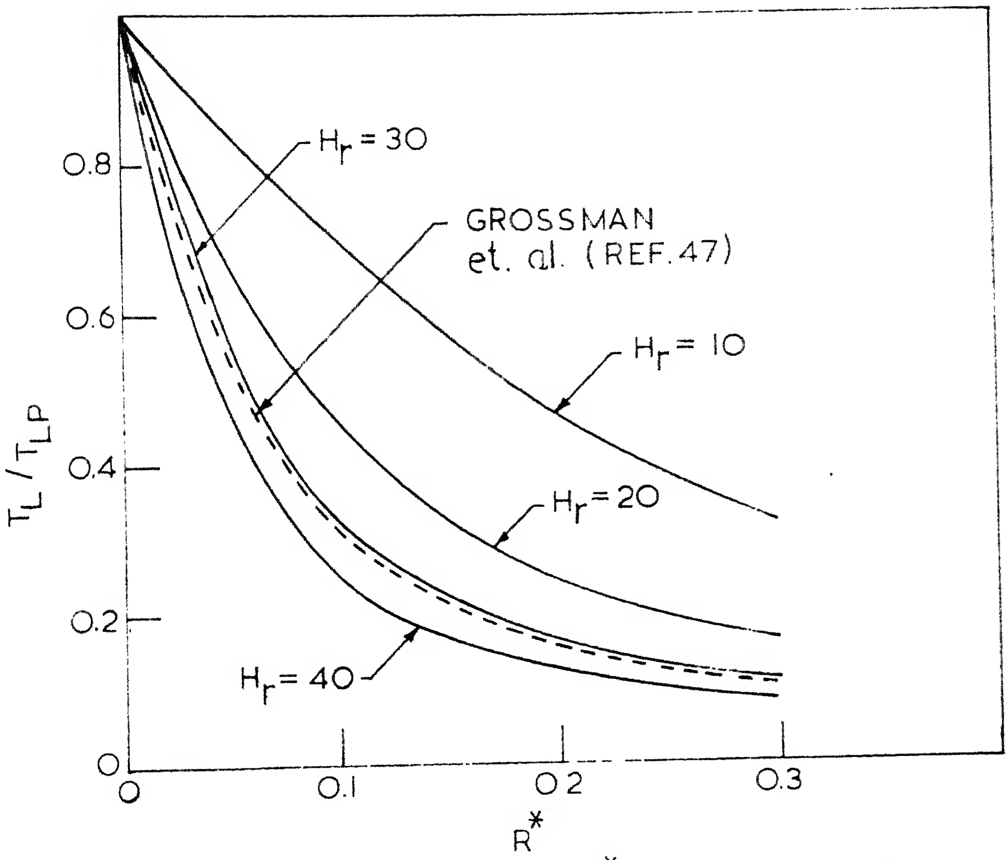

FIG. 19 – T_L' / T_{LP} VERSUS R^* FOR A SPHERICAL CAP (T_{LP} DENOTES CORRESPONDING T_L FOR CIRCULAR PLATE)

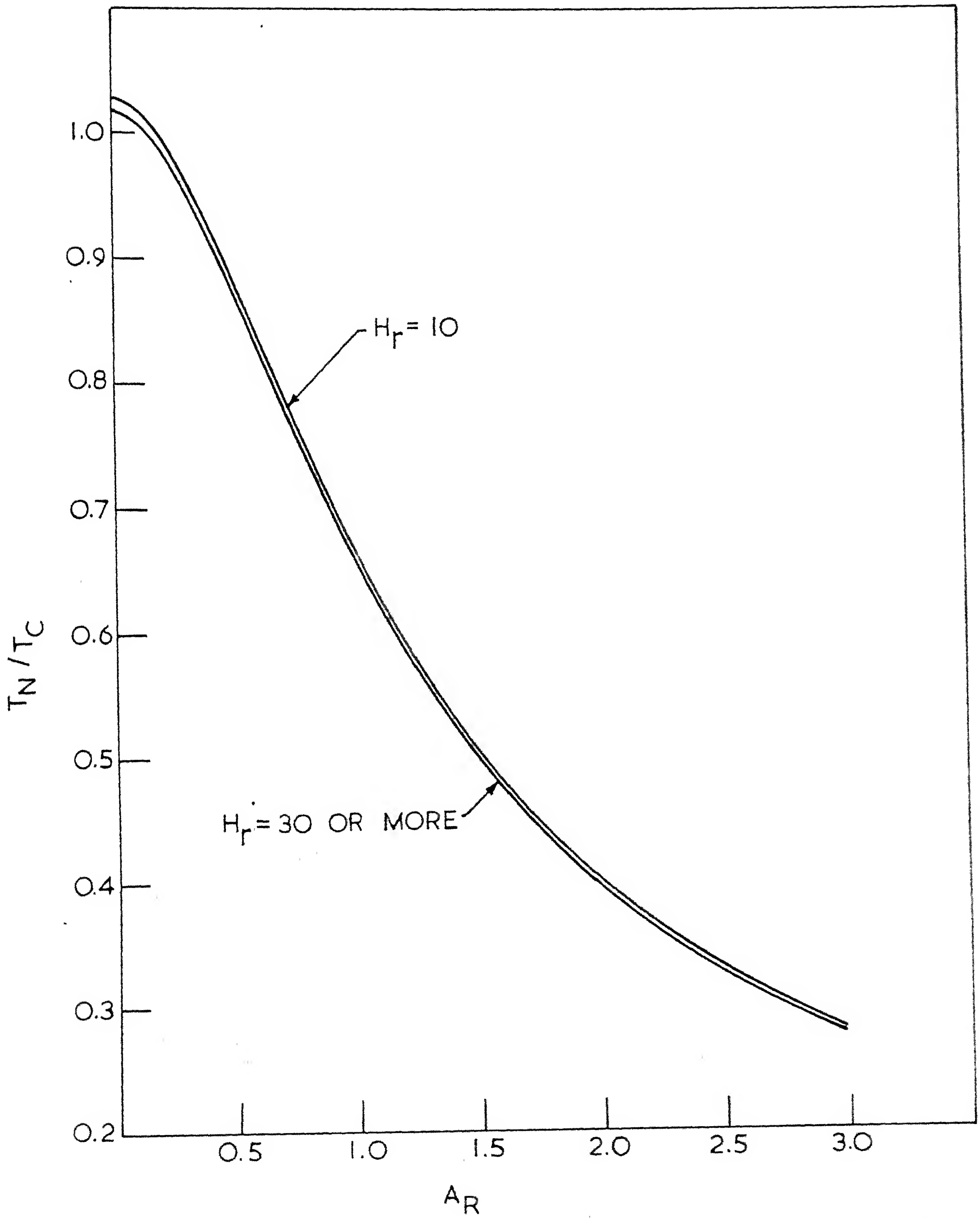

FIG. 20 – T_N/T_C VERSUS A_R FOR A CIRCULAR PLATE

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